

# Generic method for bijections between blossoming trees and planar maps

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## Abstract

This article presents a unified bijective scheme between planar maps and blossoming trees, where a blossoming tree is defined as a spanning tree of the map decorated with some dangling half-edges that enable to reconstruct its faces. Our method generalizes a previous construction of Bernardi by loosening its conditions of applications so as to include *annular maps*, that is maps embedded in the plane with a root face different from the outer face.

The bijective construction presented here relies deeply on the theory of  $\alpha$ -orientations introduced by Felsner, and in particular on the existence of minimal and accessible orientations. Since most of the families of maps can be characterized by such orientations, our generic bijective method is proved to capture as special cases all previously known bijections involving blossoming trees: for example Eulerian maps,  $m$ -Eulerian maps, non separable maps and simple triangulations and quadrangulations of a  $k$ -gon. Moreover, it also permits to obtain new bijective constructions for bipolar orientations and  $d$ -angulations of girth  $d$  of a  $k$ -gon.

As for applications, each specialization of the construction translates into enumerative by-products, either via a closed formula or via a recursive computational scheme. Besides, for every family of maps described in the paper, the construction can be implemented in linear time. It yields thus an effective way to encode and generate planar maps.

In a recent work, Bernardi and Fusy introduced another unified bijective scheme, we adopt here a different strategy which allows us to capture different bijections. These two approaches should be seen as two complementary ways of unifying bijections between planar maps and decorated trees.

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# Introduction

The enumeration of planar maps was initiated in the 60's with the pioneering work of Tutte [38]. To obtain enumeration formulas for planar maps, Tutte uses some properties about their decomposition to write equations satisfied by their generating series. The equations thus obtained are quite complicated, in particular some additional parameters (known as *catalytic variables*) have usually to be introduced to write them. The work of Tutte is a computational tour de force, since he managed, in most cases, to solve these equations and to obtain closed (and particularly simple) formulas for numerous families of maps.

This method turned out to be extremely versatile and can be applied to many different models with only slight modifications. Furthermore, the structure of the equations and of their solutions is now better understood and some standard methods such as the “quadratic method” [23, sec.2.9] and its extensions [7] are available to find them in standard cases. Tutte's approach, however, gives little insight about the combinatorial structure of maps and in particular produces no explanation of why the formulas obtained should be so simple.

Since then, successful ideas have been used to rederive and generalize the results of Tutte, including work based on matrix integrals [37, 12], algebraic approach [25] and bijective constructions [15, 35]. The latter yield an elegant way to rederive the formulas of Tutte but they also provide tools to understand better the combinatorial structure of maps: they produce for example an efficient way to encode them by languages of words and hence to sample them efficiently [35, 33]. This led to establish new conjectures about the asymptotic behaviour of random maps, which gave birth to a new field of research that has been extremely active since (see for instance [14, 28, 26, 30, 27]). It is noteworthy that a key ingredient in all these works about the convergence of random planar maps is Schaeffer's bijection [35] between quadrangulations and well-labelled trees, where the distance in the quadrangulation between a vertex and the root is encoded by the label of the corresponding vertex in the tree, or one of its generalizations [10, 13, 29].

Let us now focus on those bijective proofs. Formulas obtained by Tutte and its successors suggest that maps could be interpreted as trees with some decorations. After initial work in that direction by Cori and Vauquelin [15], Schaeffer [35] drew new attention on this field by obtaining numerous bijective constructions. This founding work was followed by a series of papers dealing with various families of maps: a non-exhaustive list includes maps with prescribed degree sequence [34, 9, 10], maps endowed with a physical model [8, 11] or with connectivity constraints [33]. Each of these bijections appears as an ad-hoc explanation of the known enumeration formula, but they present strong similarities, which calls for a unified bijective theory. An important step in that direction has been achieved in [4, 5], where a “master bijection” is introduced in order to see many constructions as special cases of a common construction. The main purpose of the present paper is to present a different attempt in unifying the bijective constructions in particular so as to include some bijections that are not captured by the work of Bernardi and Fusy.

Allow us to be slightly more precise. The first bijections obtained rely on the existence

of a canonical spanning tree of the map [34, 35, 9] or of its quadrangulation [15, 35, 10]. As emphasized by Bernardi [3], a map endowed with a spanning tree can also be viewed as a map endowed with an orientation of its edges with specific properties. The latter point of view appears to be more suitable to unify and generalize the constructions. In [4] and [5], orientations are defined on the superimposition of a planar map, its dual and their (common) quadrangulation. The master bijection is based on this orientation and produces a bicolored tree where white vertices correspond to the vertices of the map and black vertices to its faces. This construction includes as special cases many previously known bijections, but unfortunately not all of them and in particular not the case of simple triangulations [33] and quadrangulations [20, ch.3] in which the tree is simply a spanning tree of the map. This is maybe the only drawback of the very general setting introduced in these papers.

The ground result of our paper is to present a new bijective scheme that relies on an orientation of solely the edges of the maps and yields a spanning tree of the map with decorations that allow to reconstruct facial cycles. It generalizes the result of [3] by loosening the rooting conditions; in particular it enables to deal with *annular maps* (that is, rooted planar maps with a marked face) such as triangulations of a  $p$ -gon [33]. Notably all the previous bijective constructions that involve a spanning tree of the map are captured by our generic scheme and, moreover, we obtain new bijections for plane bipolar orientations and  $d$ -angulations of a  $p$ -gon with girth  $d$ . Besides, the first bijective proof of a well-known theorem by Hurwitz on products of transpositions in the symmetric group has been obtained recently by Duchi, Schaeffer and the second author [17] using this generic scheme.

Bijjective proofs appear often as an a posteriori enlightening explanation of a simple enumerative formula. In fact, the formula is used as a guide to construct the “simplest” objects that it enumerates: the right objects to consider can be seen as its combinatorial translation. Here, remarkably, the satisfying orientations are natural enough so that they can be guessed even if a formula is not available. This is in particular the case for  $d$ -angulations, for which no closed enumeration formula is known.

Another important feature of our generic scheme is its constructive character: given a blossoming tree, the corresponding map can be computed in linear time. Reciprocally given a map endowed with the appropriate orientation, the corresponding blossoming tree can be computed in quadratic time by a generic algorithm. In fact, for all the families of maps considered in this paper, ad-hoc algorithms can be designed to compute the blossoming tree in linear time. This was known to be true for rooted maps [3], and one of our main contribution is to design a linear-time algorithm that computes the blossoming tree of a  $d$ -angulations of a  $p$ -gon.

To conclude, let us mention three perspectives to continue this work. The bijective method we develop relies deeply on orientations and, algorithmically speaking, takes as input a map endowed with a specific orientation. An algorithm by Felsner [18] (see also [20, p.56]) ensures that, for a fixed map and a prescribed sequence  $\alpha$  of outdegrees, an

$\alpha$ -orientation (if any exists) can be computed by a generic algorithm of complexity  $n^{3/2}$  (if the map has  $n$  vertices with bounded maximal outdegree). For various families of maps, linear time algorithms do exist, but it is still an open problem to design such an algorithm for  $d$ -angulations when  $d \geq 5$  (unlike simple triangulations and quadrangulations, see for instance [20, ch.2]).

Secondly, almost all models of maps appear now as special cases either of our generic scheme or of the master bijections of Bernardi and Fusy. Nevertheless, a few models are still not captured; this in particular the case of model with “matter” such as the Ising model, for which some bijections with blossoming trees are known [8]. Some additional work is needed to either generalize one of those schemes or to come up with an alternative approach.

Lastly, as mentioned above, the scaling limit of random plane maps has been a very active area of research in the last years. So far, it has been proved that for  $p = 3$  or  $p$  even, the limit of (properly rescaled)  $p$ -angulations is the so-called “Brownian map” [30, 27]. It is widely believed that all the reasonable families of maps – which includes for instance  $p$ -angulations for  $p$  odd or maps with constraints on their connectivity – belong to the same universality class or in other words should converge to the same limit object. A first result in this direction about simple triangulations and quadrangulations has been obtained very recently by Addario-Berry and the first author [1]. The proof of their result relies on the bijections of [33] and [20] and on a way to interpret the distances in the map on the corresponding blossoming tree. It would be a major breakthrough to generalize their result to all the maps captured by our scheme.

**Outline** In Section 1, we gather definitions about maps and orientations and recall the fundamental result of Felsner about uniqueness of minimal  $\alpha$ -orientations (Theorem 1.1). We introduce *blossoming maps* in Section 2 and describe and prove our bijective scheme along with some remarks about its complexity.

In Section 3, previous bijections obtained for Eulerian maps and general maps (Subsection 3.1),  $m$ -Eulerian maps (Subsection 3.2) and non separable maps (Subsection 3.3) are rederived via our bijective technique. A new bijection between bipolar orientations and some triples of paths is obtained in Section 4.

Sections 5 and 6 are devoted to  $d$ -angulations of a  $p$ -gon. More precisely, Section 5 describes  $\frac{d}{d-2}$ -orientations and the bijection between  $p$ -gonal  $d$ -fractional forests and  $p$ -gonal  $d$ -angulations as well as enumerative consequences, while Section 6 focuses on the description and the proof of the linear time opening algorithm in that setting.

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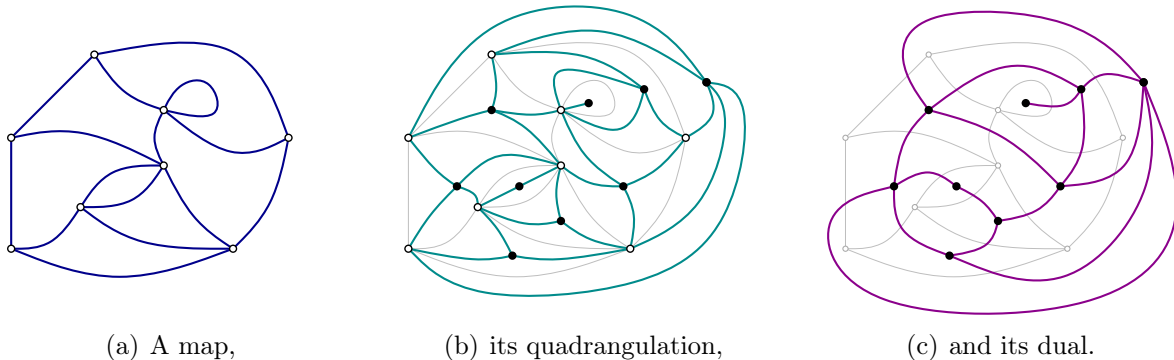


Figure 1: Example of the quadrangulation and the dual of a map.

# 1 Maps and orientations

## 1.1 Planar maps

A *planar map* is a proper embedding of a connected graph in the sphere, where *proper* means that edges are smooth simple arcs which meet only at their endpoints. Two planar maps are identified if they can be mapped one onto the other by a homeomorphism that preserves the orientation of the sphere. *Edges* and *vertices* of a map are the natural counterparts of edges and vertices of the underlying graph. The *faces* of a map are the connected components of the complementary of the embedded graph. The embedding fixes the cyclical order of (half-)edges around each vertex, which defines readily a *corner* as a couple of consecutive (half-)edges around a vertex (or, equivalently, around a face); corners may also be viewed as incidences between vertices and faces. The *degree* of a vertex or a face is defined as the number of its corners. In other words, it counts incident edges with multiplicity 2 for each loop (for vertex degree) or bridge (for face degree).

In these definitions, vertices and faces play a similar role; it is often useful to exchange them and to consider the *dual*  $M^*$  of a given map  $M$ , whose vertices correspond to faces of  $M$  and faces to vertices of  $M$ . Edges are somehow unchanged: each edge  $e$  of  $M$  corresponds to an edge of  $M^*$  that is incident to the same vertices and faces as  $e$ , see Fig.1(c).

A planar map is said to be *d-regular* if all its vertices have degree  $d$ . Dually, a planar map is called a *d-angulation* if all its faces have degree  $d$ ; the terms *triangulation*, *quadrangulation* and *pentagulation* correspond respectively to the cases where  $d = 3, 4, 5$ . A useful construction associates a quadrangulation with each planar map  $M$ : let us say that vertices of  $M$  are white; let us add a (black) vertex in each face of  $M$ , and an edge in each corner of  $M$  between the corresponding white and black vertices. This produces a triangulation with bicolored vertices; keeping only the additional edges leads to a quadrangulation  $\mathcal{Q}_M$  that is called *the quadrangulation of  $M$* , see Fig.1(b). Its dual  $\mathcal{R}_M$  is called the *radial map* of  $M$ .

A *plane map* is a proper embedding of a connected graph in the plane, its unique

unbounded face is called the *outer face*, all the other faces are called *inner faces*. Vertices and edges are called *outer* or *inner* depending on whether they are incident to the outer face or not. Observe that a plane map is in fact a planar map with a distinguished marked face.

A planar or plane map is said to be *(corner)-rooted* if one corner is distinguished. The corresponding vertex and face are called *root vertex* and *root face*, the *root edge* is defined as the edge that follows the root corner in clockwise order around the root vertex. The usual convention is to associate to each rooted planar map the rooted plane map in which the root and the outer faces coincide. However in this work, plane maps are allowed to have one root face different from the outer face (in the literature, planar maps for which the root face is different from the outer face are sometimes called *annular maps*). Some weaker rootings will sometimes be also considered by only pointing either a root vertex, a root edge or a root face. In the latter case, vertices or edges incident to the root face are called root vertices or root edges.

A *plane tree* is a planar map with a single face; its vertices are called *leaves* and *nodes* depending on whether their degree is equal to one or not. A *planted tree* is a plane tree rooted at a leaf. Observe that usual “planar trees” or “ordered trees” are obtained from planted trees by deleting their root leaf.

## 1.2 Orientations

This section gathers definitions and fundamental results about orientations of planar maps. The terminology and convention are not completely standard and we emphasize the differences when needed.

An *orientation* of a map is the choice of a direction for each of its edges. The *indegree* or *outdegree* of a vertex  $v$ , denoted  $\text{in}(v)$  or  $\text{out}(v)$ , is the number of edges oriented inwards or outwards  $v$ . Let  $M$  be a planar map,  $V$  the set of its vertices, and let  $\alpha : V \rightarrow \mathbb{N}$  be an application which associates a natural number to each vertex of the map. An  $\alpha$ -*orientation* – as introduced by Felsner in [18] – is an orientation of  $M$  such that for each vertex  $v$  in  $V$ ,  $\text{out}(v) = \alpha(v)$ . If such an orientation exists,  $\alpha$  is said to be *feasible*. An orientation of a corner-, vertex- or face-rooted map is said to be *accessible* if for any vertex  $v$ , there exists a directed path from  $v$  to the root vertex (or to one of the root vertices, in the case of a face rooting).

An *oriented path* is an alternating sequence  $(v_0, e_1, v_1, \dots, v_{\ell-1}, e_{\ell}, v_{\ell})$  of incident vertices and edges in which each edge  $e_i$  is oriented from  $v_{i-1}$  to  $v_i$ . An *oriented cycle* (also called a *circuit*) is defined accordingly. The (canonical) embedding of plane maps enables to define *clockwise cycles* and *counter-clockwise* or *direct cycles* as simple oriented cycles with the outer face respectively on their left or on their right. Observe that the orientation obtained after reverting the direction of all the edges of a given oriented cycle is still an  $\alpha$ -orientation. In fact, all the  $\alpha$ -orientations of a map  $M$  can be obtained by a sequence of such *flips*, see [18]. In particular, it implies that either all or none  $\alpha$ -orientations of  $M$  are accessible. In the former case,  $\alpha$  is said *accessibly feasible*. Moreover:

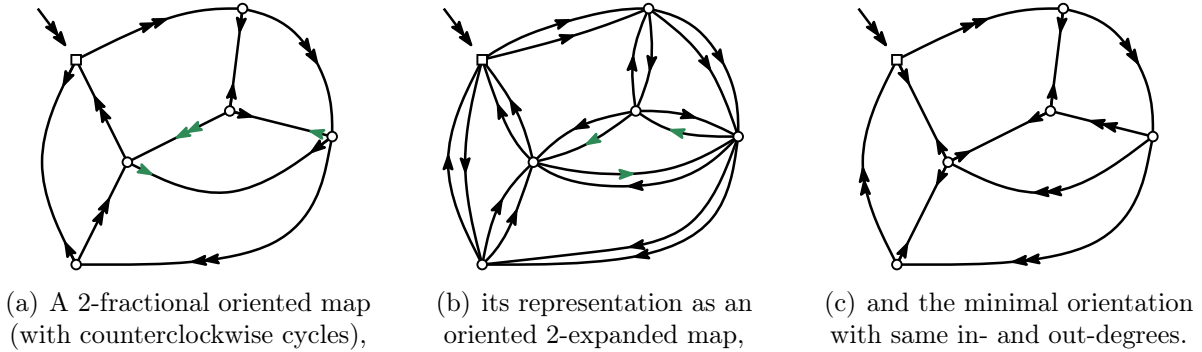


Figure 2: A rooted plane map endowed with accessible 2-fractional orientations.

**Theorem 1.1** (Felsner [18]). *Let  $M$  be a plane map and  $\alpha$  be a feasible function on its vertices. The set of  $\alpha$ -orientations of  $M$  can be endowed with a lattice structure, whose minimal element is the unique  $\alpha$ -orientation of  $M$  without counterclockwise cycles.*

The main consequence of this theorem for our purpose is to associate canonically to any given feasible  $(M, \alpha)$  the (unique) minimal  $\alpha$ -orientation of  $M$ . From now on, we call *minimal* any orientation without counterclockwise cycles.

The  $k$ -expanded version of a plane map  $M$  is defined as the plane map where each edge of  $M$  has been replaced by  $k$  copies. A  $k$ -fractional orientation of  $M$  is defined in [4] as an orientation of the  $k$ -expanded map of  $M$ , with the additional property that two copies of the same edge cannot create a counterclockwise cycle. It is conveniently considered as an orientation of  $M$  in which edges can be partially oriented in both directions and the in- or out-degree of a vertex  $v$  (that can now be fractional) is defined as the in- or out-degree of  $v$  in the  $k$ -expanded map, divided by  $k$ . In this setting, a *saturated edge* is an edge which is totally oriented in the same direction, a *directed path* is a path in which each edge is at least partially oriented in the considered direction. The notions of clockwise or counterclockwise cycles, of minimality and of accessibility follow.

## 2 A generic bijective scheme for maps endowed with a minimal orientation

### 2.1 Blossoming trees and maps and closure

**Definition 2.1.** *A blossoming map is a plane map, in which each outer corner can carry a sequence of opening or closing stems (in the literature, opening and closing stems are sometimes referred to as buds and leaves).*

*The cyclic contour word of a blossoming map is the word on  $\{e, b, \bar{b}\}$ , which encodes the cyclic clockwise order of edges and stems along the border of the outer face with  $e$  coding for an edge and  $b$  and  $\bar{b}$  for opening and closing stems.*

A *local closure* is a substitution in the contour word of a factor  $be^*\bar{b}$  by the letter  $e$ , where  $e^*$  denotes any sequence of  $e$  (possibly empty). In terms of maps, it corresponds to the creation of a new edge (and hence a new face) by merging an opening stem with the following closing stem (provided that there are no other stem in between) in clockwise order around the border of the outer face; the new edge is canonically oriented from the opening vertex to the closing vertex, with the new bounded face on its right.

**Definition 2.2.** *The closure  $\overline{M}$  of a blossoming map  $M$  is the (possibly blossoming) map obtained after iterating as many local closure operations as possible. When only a subset of local closures is performed, the map obtained is called a partial closure of  $M$ .*

*The edges created during local closures operations are called closure edges.*

In particular, the closure of a blossoming map with an equal number of opening and closing stems is a (standard, non-blossoming) map. Since closure edges are canonically oriented, if a blossoming map is endowed with an orientation (possibly  $k$ -fractional), so is its closure. Moreover, considering opening and closing stems respectively as outgoing and incoming (half-)edges, in- and out-degrees are preserved. Since all the closures are performed in clockwise direction around the map, no counter clockwise cycle can be created during a local closure operation. Consequently if the initial orientation is minimal, then so is its closure. Accessibility is conserved as well.

The most interesting special case is the one of a rooted plane tree, endowed with an accessible orientation, which, in the classical non-fractional setting, implies that edges are oriented towards the root vertex. Examples of a blossoming tree and of its closure are given in Figs. 3(c) and 3(a). See also Fig. 6(c) and 6(a) for a 2-fractional example.

The aim of the next section is to provide an inverse construction of the closure.

## 2.2 Orientations and opening

The following theorem generalizes a result on tree orientations that can be explicitly found *e.g.* in [3], and which is at the heart of all bijections between map orientations and blossoming trees.

**Theorem 2.3.** *Let  $M$  be a plane map vertex-rooted at  $r$ , and suppose that  $M$  is endowed with a minimal accessible orientation  $O$ . Then  $M$  admits a unique edge-partition  $(\mathcal{T}_M, \mathcal{C}_M)$  such that:*

- *edges in  $\mathcal{T}_M$  (called tree edges) form a spanning tree of  $M$ , rooted at  $r$ , on which the restriction of  $O$  is accessible;*
- *any edge in  $\mathcal{C}_M$  (called a closure edge) is a saturated clockwise edge in the unique cycle it forms with edges in  $\mathcal{T}_M$ .*

*Let us call such a partition a tree-and-closure partition.*



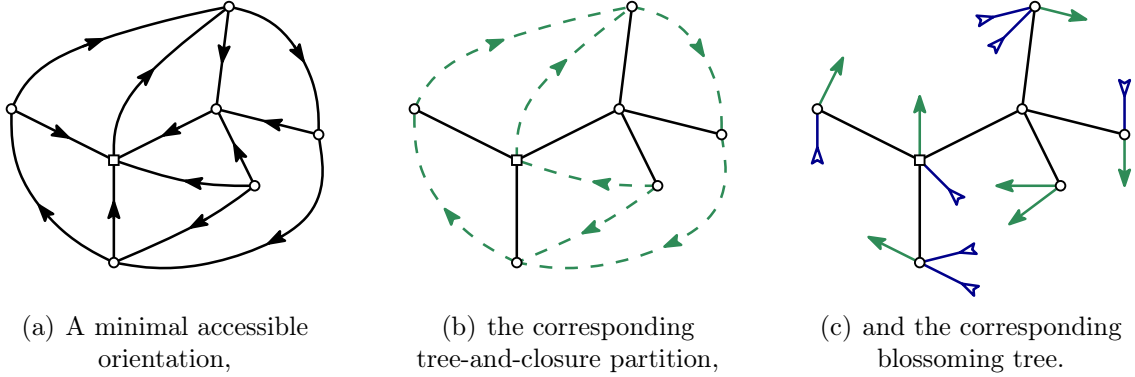


Figure 3: From a minimal accessible orientation to a blossoming tree.

Before proving this theorem, we would like to emphasize the difference with the result obtained in [3]. In the latter work, the outer face of a rooted map is required to be the root face. In this case and also in the particular case of triangulations treated in [33], a contour algorithm, starting at the root edge, enables to identify the edges of  $\mathcal{C}_M$ . The proof of this algorithm relies deeply on the fact that *both* the accessibility and the minimality of the orientation are defined according to the root face. We show here that this hypothesis is unnecessary.

*Proof.* We prove this result by induction on the number of faces of  $M$ . If  $M$  has only one face, it is an accessible tree, hence the property is satisfied.

Let now  $n \geq 2$ , and suppose that the property is satisfied for any plane map with strictly less than  $n$  faces. Let  $M$  be a vertex-rooted plane map with  $n$  faces endowed with a minimal accessible orientation.

We shall first prove that one of its outer edges  $e$  may be removed to obtain a vertex-rooted map  $M_{\setminus e}$  endowed with a minimal accessible orientation. As no cycle of  $M$  is oriented counterclockwise, there exists at least one (saturated) outer edge  $(u, v)$  that is not a bridge and has the outer face on its left. Let us consider the map  $M_{\setminus (u, v)}$ . If it is accessible, then choose  $e = (u, v)$ . If not, let  $C$  be the accessible component of  $r$  in  $M_{\setminus (u, v)}$ . Then clearly  $v$  belongs to  $C$  and  $u$  does not, and  $u$  is accessible from all vertices not in  $C$ , see Fig. 4(a). Let  $D$  be the accessible component of  $u$ . Since  $u$  and  $v$  both are incident to the outer face of  $M$ , and  $(u, v)$  is not a bridge, the cut between  $C$  and  $D$ , made up of saturated edges oriented from  $C$  to  $D$ , is incident twice to the outer face of  $M_{\setminus (u, v)}$ . Let  $e$  be the edge of the cut with the outer face on its left. Then  $e$  is not a bridge in  $M$ , hence  $M_{\setminus e}$  has  $n - 1$  faces, and the orientation induced by that of  $M$  is minimal and accessible.

Hence, by induction,  $M_{\setminus e}$  admits a (unique) tree-and-closure partition  $(\mathcal{T}_M, \mathcal{C}_M)$ , and  $(\mathcal{T}_M, \mathcal{C}_M \cup \{e\})$  is a tree-and-closure partition for  $M$ .

Let us now prove that  $M$  does not admit any other tree-and-closure partition, that is, does not admit any tree-and-closure partition with  $e$  in the tree. Suppose by contradiction that  $(\mathcal{T}'_M, \mathcal{C}'_M)$  is a tree-and-closure partition for  $M$  with  $e \in \mathcal{T}'_M$ . Let us denote  $e = (x, y)$ , oriented from  $x$  to  $y$ , and consider the simple path  $\gamma$  from  $x$  to the root vertex in  $\mathcal{T}_M$ .

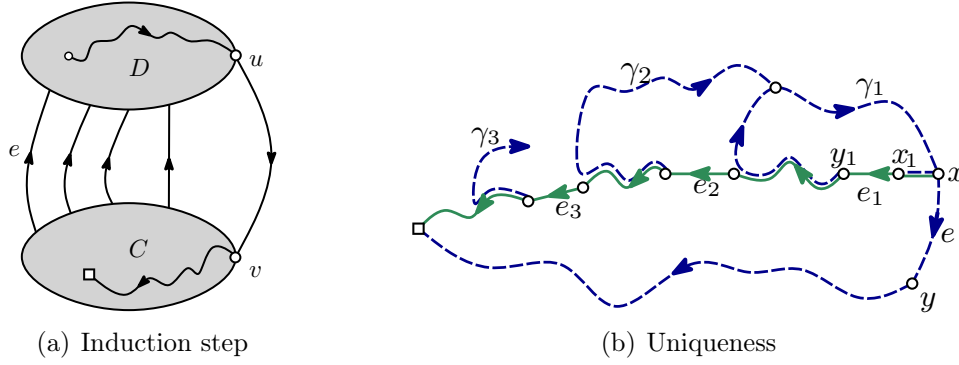


Figure 4: Existence of a unique tree-and-closure partition.

At least one edge of  $\gamma$  does not belong to  $\mathcal{T}'_M$  (otherwise this would create a cycle in the tree), hence  $\gamma$  contains saturated edges, and in particular the first one,  $e_1 = (x_1, y_1)$ , cannot belong to  $\mathcal{T}'_M$  since  $x_1$  lies in the subtree  $\mathcal{T}'_M(x)$  of  $\mathcal{T}'_M$  rooted at  $x$ . Hence  $e_1$  turns clockwise around the cycle  $\gamma_1$  it forms with  $\mathcal{T}'_M$ , and since  $(x, y)$  is an outer edge,  $y_1$  also belongs to  $\mathcal{T}'_M(x)$ , see Fig. 4(b). In particular, there exists another (saturated) edge  $e_2$  of  $\gamma$  that does not belong to  $\gamma_1$  nor  $\mathcal{T}'_M$  for which the same reasoning applies. This implies the existence of an infinite sequence of edges of  $\gamma$  not belonging to  $\mathcal{T}'_M$ , contradiction.  $\square$

As an immediate corollary of Theorem 2.3, we obtain:

**Corollary 2.4.** *Let  $M$  be a vertex-rooted plane map endowed with a minimal accessible orientation  $O$ . Then there exists a unique vertex-rooted blossoming tree, endowed with an accessible orientation, the closure of which is  $M$  oriented with  $O$ .*

*This blossoming tree is denoted  $\mathcal{B}_M$ .*

Hence, in any particular case where a family of planar maps may be canonically endowed with a specific minimal and accessible orientation, Corollary 2.4 gives a bijection between that family and a family of blossoming trees with the same distribution of in- and out-degrees. If these trees are easily described and enumerated, this can lead to a bijective proof of enumerative results.

## 2.3 Effective opening and closure

Let us point out some facts about the complexity of computing effectively the closure of a blossoming tree and the opening of a map.

To close a blossoming tree into a map it is enough to perform a contour process and to match iteratively each opening stem with its corresponding closing stem along the way. Each time a new opening stem is explored, it can be stored in a Last-In-First-Out stack out of which one will be popped each time a closing stem is explored. This process goes around the outer face at most twice, hence the total time complexity is linear in the number of edges of the final map.

Unfortunately, things are not so smooth when it comes to opening an oriented plane map into its blossoming tree. Since the proof of Theorem 2.3 is essentially constructive, it yields an algorithm which identifies a closure edge at each step. Each of them consists in computing an accessible component, which can be done in linear time, resulting in a total quadratic complexity.

However, in the case where the map is corner-rooted in the outer face, the opening operation can be realized in linear time by an adapted depth-first search process. This construction has been introduced in a series of papers (see for example [35, 33]) in some particular cases and formally stated in [3] (where it appears in a slightly different form since the convention for tree edges orientation is opposite to ours).

**Proposition 2.5** ([35, 33, 3]). *Let  $M$  be a corner-rooted plane map in which the outer and root faces coincide, and assume that  $M$  is endowed with a minimal orientation.*

*Then, the tree-and-closure partition of  $M$  can be computed in linear time: initialize  $\mathcal{T}_M$  and  $\mathcal{C}_M$  as empty sets, and  $v$  and  $e$  to be respectively the root vertex and the root edge; then repeat the following steps until all edges belong either to  $\mathcal{C}_M$  or to  $\mathcal{T}_M$ :*

- *if  $e$  does not belong to  $\mathcal{C}_M$  nor  $\mathcal{T}_M$  yet, add it to  $\mathcal{C}_M$  if it is oriented outwards  $v$ , and to  $\mathcal{T}_M$  otherwise;*
- *if  $e$  belongs to  $\mathcal{T}_M$ , switch  $v$  to the other extremity of  $e$ ;*
- *update  $e$  to the next edge around  $v$  in clockwise order.*

The proof of this proposition can be found in [3], in a different setting: it deals with rooted planar maps endowed with a distinguished spanning tree, for which an orientation is canonically defined by orienting tree edges towards the root, and any other clockwise in the unique cycle it forms with the tree. The set of all these *tree orientations* is hence equal to the set of minimal  $\alpha$ -orientations for all accessibly feasible  $\alpha$ .

Observe that as opposed to the purpose of [3], our work aims at defining one canonical orientation for each map. This requires to seek for an appropriate family of functions  $\alpha$  for each family of maps we want to enumerate, as already mentioned at the end of Section 2.2. In the next section, we demonstrate that numerous previously known bijections can be easily retrieved as soon as we exhibit the adequate  $\alpha$ .

### 3 Recovering previous bijections

Previous bijections between planar maps and blossoming trees obtained after Schaeffer [34] can all be seen as applications of Corollary 2.4, and more precisely of Proposition 2.5. In this section, planar maps are corner-rooted with the root corner in the outer face. Hence the blossoming trees involved are *balanced*, meaning that no local closure may wrap the root corner. More formally, let us define the (non cyclic) *contour word* of a rooted blossoming tree as the natural counterpart of the cyclic contour word starting at the root corner, see

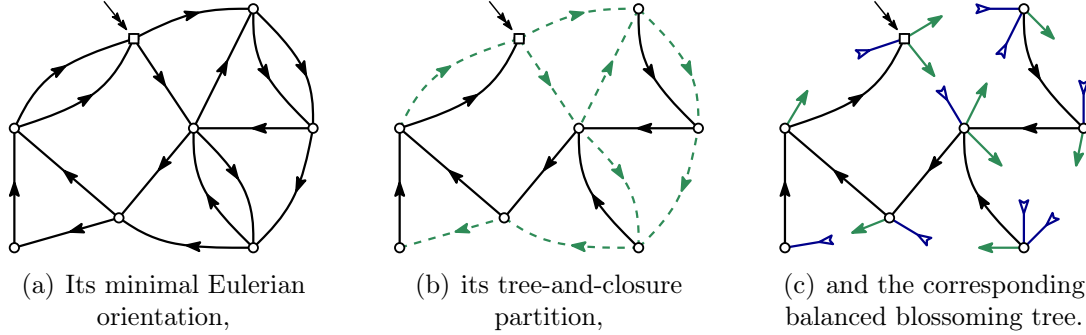


Figure 5: Blossoming trees for an Eulerian map.

Definition 2.1. Then a rooted blossoming tree is said *balanced* if the restriction of its contour word on  $\{b, \bar{b}\}$  is a Dyck word.

The intuition behind these bijections was originally relying on the interpretation of the enumerative formulas. We want to emphasize here that most of the time, a natural choice for a function  $\alpha$  leads to the same construction. We only sketch how to retrieve the proofs that can be found in the original papers.

### 3.1 Maps with prescribed vertex degree sequence

**Eulerian maps** The first bijection obtained by Schaeffer in [34] concerns *Eulerian maps* with prescribed vertex degrees, and in particular 4-regular maps with  $n$  vertices, that correspond bijectively to planar maps with  $n$  edges.

This bijection can be recovered in the following way. First recall that a map is said *Eulerian* if its vertices have even degrees. It is a classical result that any Eulerian map may be endowed with orientations with equal in- and out-degrees for each vertex, and these are accessible. In particular, the minimal *Eulerian orientation* of a given plane Eulerian map can be obtained recursively by orienting clockwise the outer cycle and erasing it, see Fig. 5(a).

The generic opening of a planar rooted Eulerian map  $M$  with  $n_i$  vertices of degree  $2i$  for any  $i \in \llbracket 1, k \rrbracket$  (and hence  $n = \sum_i i n_i$  edges) endowed with its minimal Eulerian orientation leads to a balanced rooted blossoming tree with the same distribution of vertex in- and out-degrees, and both  $\ell = 1 + \sum_i (i-1)n_i$  opening and closing stems. Observe that each non-root vertex has exactly one outgoing edge that belongs to the blossoming tree; moreover, since the tree is balanced, the root corner is necessarily followed by an opening stem.

To enumerate balanced blossoming trees, a general strategy is to consider a larger family of planted blossoming trees that is stable by rerooting and in which the proportion of balanced ones can be evaluated. Let us sketch this strategy in the case of eulerian maps, following [34]. Let us first consider planted trees with  $n_i$  nodes of degree  $i+1$  (*i.e.* arity  $i$ ) for any  $i > 0$  and hence  $\ell + 1 = 2 + \sum_i (i-1)n_i$  leaves *including the root one*; they are

enumerated by ([24]):

$$T_{n_1, \dots, n_k} = \frac{1}{n} \binom{n}{\ell, n_1, \dots, n_k} = \frac{(n-1)!}{\ell!} \prod_{i=1}^k \frac{1}{n_i!}.$$

Start from one such tree and add  $(i-1)$  opening stems on each node of arity  $i$ . The total number of trees that can be obtained in this way is then:

$$B_{n_1, \dots, n_k} = \prod_{i=1}^k \binom{2i-1}{i}^{n_i} T_{n_1, \dots, n_k} = \frac{(n-1)!}{\ell!} \prod_{i=1}^k \binom{2i-1}{i}^{n_i} \frac{1}{n_i!}.$$

Consider now each leaf (including the root one) as a closing stem, it yields a blossoming tree with  $\ell-1$  opening stems and  $\ell+1$  closing stems, whose closure gives a map with two unmatched closing stems.

Among all the planted blossoming trees that give the same map, a proportion  $2/(\ell+1)$  of them are rooted on one of the unmatched stems. For those ones, changing their root (closing) stem into an opening one leads to a balanced blossoming tree. Hence:

**Proposition 3.1. (Eulerian planar maps with prescribed vertex degrees)** *The number of rooted planar Eulerian maps with  $n_i$  vertices of degree  $2i$  for any  $i \in \llbracket 1, k \rrbracket$  is given by:*

$$\frac{2 \cdot (n-1)!}{(\ell+1)!} \prod_{i=1}^k \binom{2i-1}{i}^{n_i} \frac{1}{n_i!}.$$

An interesting particular case concerns rooted planar 4-regular maps with  $n$  vertices, that correspond bijectively to rooted planar maps with  $n$  edges. Indeed, let  $M$  be a planar map and consider its radial map  $\mathcal{R}_M$ ; by convention,  $\mathcal{R}_M$  is rooted with the same root face as  $M$ , and its root vertex corresponds to the root edge of  $M$ .

It is clear from its definition that  $\mathcal{R}_M$  is a 4-regular map and that reciprocally every rooted 4-regular map with  $n$  vertices corresponds to a unique rooted planar map with  $n$  edges. According to our generic bijective scheme, rooted 4-regular maps are in bijection with balanced planted blossoming trees with  $n$  nodes, of in- and out-degrees 2, that is obtained from a planted binary tree by adding one opening stem to each node. Hence:

**Corollary 3.2. (Planar maps with prescribed number of edges)** *The number of rooted planar maps with  $n$  edges is:*

$$\frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}.$$

**General maps** In [9], the bijection for Eulerian maps is generalized into a bijection between planar maps with prescribed vertex degree sequence and some blossoming trees. We sketch in this paragraph how this construction can be derived from our generic scheme.

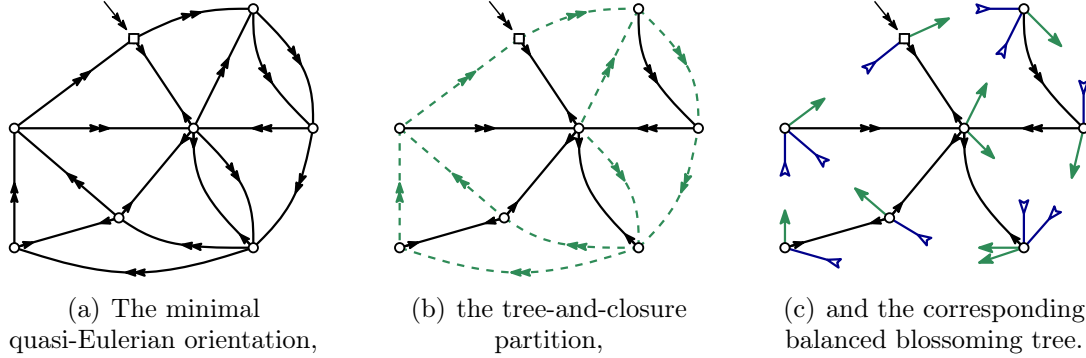


Figure 6: Blossoming trees for general maps: the generic R-case.

Any maps may be endowed with a *quasi-Eulerian* orientation, that is, a partial orientation (or a 2-fractional orientation) with equal in- and out-degrees for each vertex. In particular, as minimal Eulerian orientations, minimal quasi-Eulerian orientations can be obtained recursively; orient clockwise the outer cycle (with the convention that an edge adjacent twice to the outer face is partially oriented in both directions) and iterate after erasing outer edges (see Fig. 6(a) and 7(a)).

Let  $M$  be a rooted map endowed with its minimal quasi-Eulerian orientation. Opening  $M$  gives a rooted balanced blossoming tree endowed with an accessible 2-fractional orientation such that the in- and out-degrees of each vertex are equal. Now, to characterize blossoming trees that admit such a 2-fractional orientation, it is convenient to follow [9] and introduce the notion of *charge* of a subtree as the difference between the numbers of its closing stems and opening stems. It is then easily seen that a blossoming tree with total charge 0 admits an orientation with equal in- and out-degrees at each vertex if and only if its proper subtrees all have charge 0 or 1. More precisely, subtrees planted at a saturated edge have charge 1, while those planted at a bi-oriented edge have charge 0. This corresponds respectively to the R- and S-trees in [9].

In particular, in the generic case where the root edge  $e$  of  $M$  is not an isthmus, it is oriented clockwise and thus belongs to the closure, therefore  $\mathcal{B}_M$  consists of an R-tree and an opening stem situated right after the root corner (see Fig. 6). In the special case where  $e$  is an isthmus, it is partially oriented and belongs to the blossoming tree  $\mathcal{B}_M$ . Therefore the right subtree of  $\mathcal{B}_M$  is an S-tree carried by  $e$  and the other subtrees also form an S-tree (see Fig. 7).

Let us mention that the enumeration of these trees is not straightforward (and is carried out in Section 3 of [9]). The main difficulty comes from the fact that R-trees are not stable by rerooting.

### 3.2 $m$ -Eulerian maps with prescribed degree sequence

In [8], the authors define  *$m$ -Eulerian* maps as bipartite maps such that black vertices all have degree  $m$  and each white vertex has degree multiple of  $m$ . We give in this section a

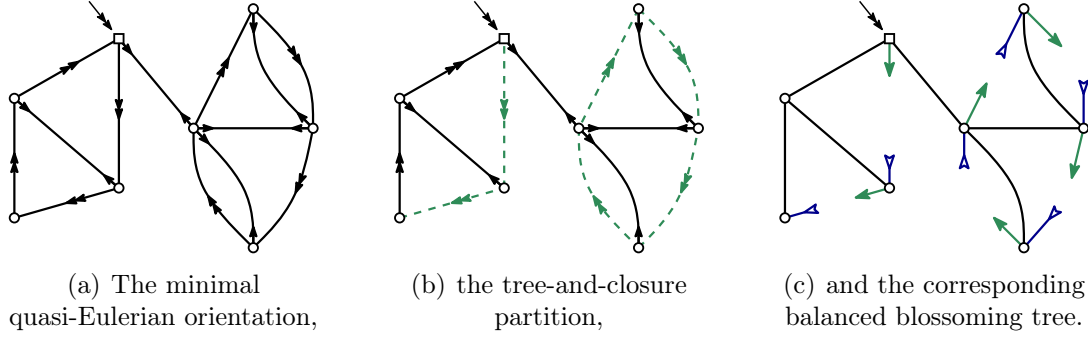


Figure 7: Blossoming trees for general maps: the isthmus S-case.

much shorter proof that these maps are in bijection with the so-called *m-Eulerian trees*. Considering black vertices as hyper-edges, *m-Eulerian* maps form in fact a subclass of *m-regular* hypermaps, which boils down to Eulerian maps if  $m = 2$ . Let us root *m-Eulerian* maps in a white corner, and call the extremities of the root edge respectively the *white* and the *black root vertex*.

An important feature of these maps is that they can be canonically labelled on edges with integers in  $\llbracket 1, m \rrbracket$  in such a way that:

- the root edge has label 1;
- around each black vertex, edges are labelled from 1 to  $m$  in clockwise order;
- around each white vertex, edges are cyclically labelled in counterclockwise order; in particular, if  $v$  is a white vertex with degree  $km$ , it is incident to exactly  $k$  edges of label  $i$  for any  $i$  in  $\llbracket 1, m \rrbracket$ .

This implies that *m-Eulerian* maps can be endowed with orientations such that each black vertex has outdegree equal to  $m - 1$  (and indegree equal to 1), and each white vertex of degree  $km$  has outdegree equal to  $k$ : just orient all edges but the 1-labelled ones from their black end to their white end, see Fig. 8(a). Moreover, this *canonical* orientation is accessible. First observe that, for any face  $f$ , labels of edges incident to  $f$  are alternatively equal to  $i$  and  $i + 1 \pmod m$  for a given  $i$ . Hence if  $i = 1$  or  $m$ ,  $f$  is an oriented cycle and all its vertices lie in the same strong connected component. For other values of  $i$ , its edges are all oriented from black to white, the  $i$ -labelled ones counterclockwise and the others clockwise. Hence, if its vertices were not in one and the same component, there would exist a cocycle which contains both an  $i$ -labelled edge  $e_0$  and an  $i + 1$ -labelled edge  $e_1$  incident to  $f$ . Iterating this argument along the cocycle produces a cyclical sequence of edges  $(e_j)$  such that the label of  $e_j$  is equal to  $i + j$ . Since a 1- or  $m$ -labelled edge cannot belong to a cocycle, we obtain a contradiction.

Now any rooted *m-Eulerian* map  $M$  with a given distribution of white vertex degrees can be canonically embedded in the plane with the root face as outer face, and endowed with its minimal  $\alpha$ -orientation with  $\alpha(v) = m - 1$  if  $v$  is black, and  $\alpha(v) = k$  if  $v$  is white

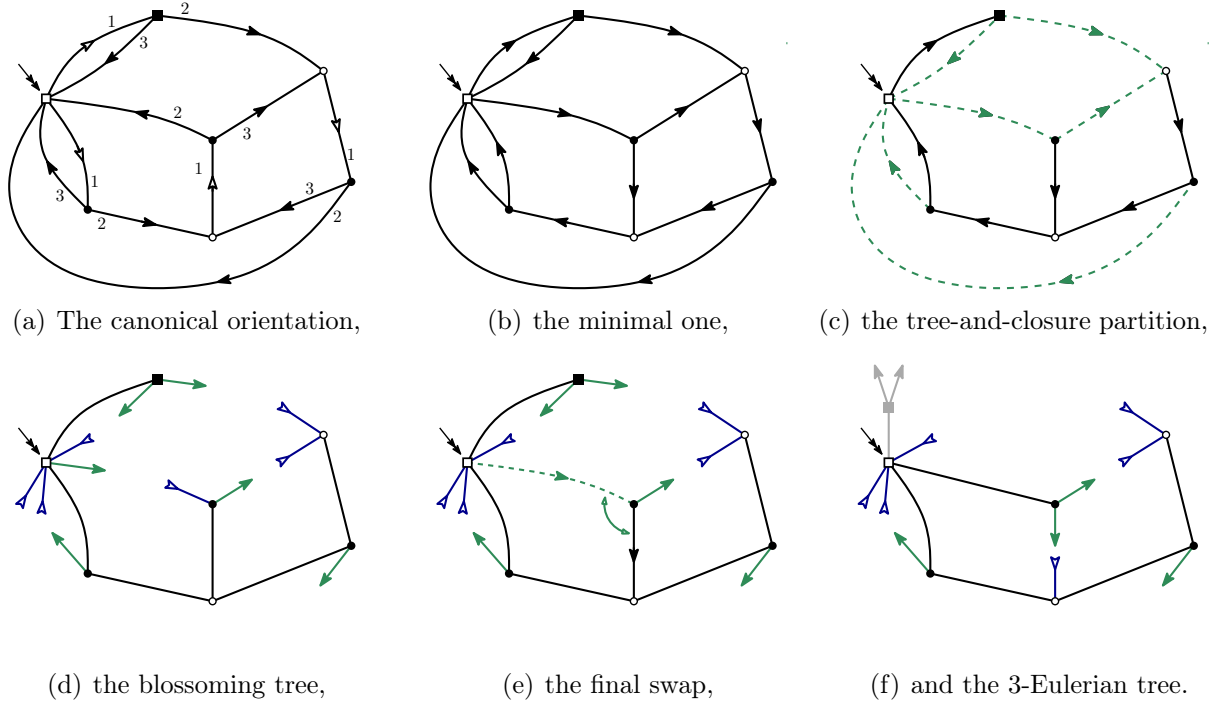


Figure 8: Blossoming trees for  $m$ -Eulerian maps: example of a 3-Eulerian map with two white vertices of degree 3 and one with degree 6, rooted on the edge with square ends.

with degree  $km$ . The generic opening according to the *black* root vertex leads to a rooted blossoming plane bipartite tree  $T$  such that (see Fig. 8(d)):

- the (black) root vertex has only one child – the white root vertex, and carries  $m - 1$  opening stems;
- any white vertex with total degree  $km$  carries  $k - 1$  opening stems, and  $k(m - 1)$  black children or closing stems;
- black non-root vertices carry  $m - 2$  opening stems and either a white child or a closing stem.

$T$  is not exactly a  $m$ -Eulerian tree as defined in [8], but the transition between the two families is easy. It is enough to modify the tree locally in a way such that black and white vertices respectively carry only opening and closing stems. To do so, observe that opening stems carried by white vertices are matched with closing stems carried by black ones (since the underlying map is bipartite). Now suppose that such a couple is carried by a white vertex  $u$  and a black vertex  $v$ . It can be replaced by its closure edge  $(u, v)$ , creating a cycle that is broken by opening the edge connecting  $v$  to its father in  $T$  so as to create a new couple of opening and closing stems (see Fig. 8(e)-(f)). This *swap* leads to a blossoming tree with only opening stems on black vertices and closing stems on white vertices; removing the black root vertex gives precisely an  $m$ -Eulerian tree. As shown in [8]:



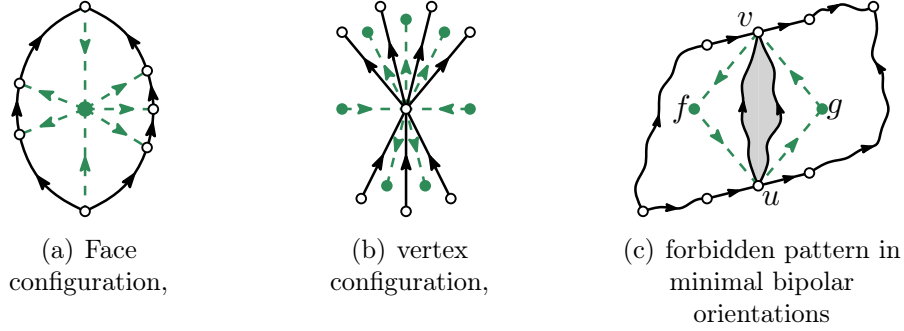


Figure 9: Properties of bipolar orientations; the non separable map is drawn in plain lines, its quadrangulation in dashed green ones.

**Proposition 3.3.** *Let  $m \geq 2$ , the number of edge-rooted  $m$ -Eulerian maps with  $d_i$  white vertices of degree  $mi$  for each  $i \geq 1$  is:*

$$m(m-1)^{v_o-1} \frac{[(m-1)v_\bullet]!}{[(m-1)v_\bullet - v_o + 2]!} \prod_{i \geq 1} \frac{1}{d_i!} \binom{mi-1}{i-1}^{d_i},$$

where  $v_\bullet = \sum id_i$  and  $v_o = \sum d_i$  denote respectively the number of black and white vertices.

### 3.3 Non separable maps with prescribed number of edges

A *cut vertex* of a map is a vertex that is incident twice to the same face; a map is said to be *non separable* if it has no cut vertex. Observe that this definition is actually stable by duality. In [34], bijections are described for general and cubic non separable planar maps, and we show here that they are indeed special cases of our generic bijective scheme.

A *bipolar orientation* of a map (or more generally a graph) is an acyclic orientation of its edges with a single *source* (vertex without any incoming edge) and a single *sink* (vertex without any outgoing edge), which are called the *poles* of the orientation. Non separable maps (or graphs) are characterized by the following property (see *e.g.* [16]):

**Proposition 3.4.** *A rooted map is non separable if and only if it can be endowed with a bipolar orientation with the two ends of the root edge as poles.*

In the planar case, rooted non separable maps endowed with a bipolar orientation (with, say, the root vertex as the sink) have some interesting properties, illustrated in Fig. 9:

1. each face is itself bipolar, hence its corners may be classified into *lateral* ones (left or right) and two *polar* ones (source and sink);
2. each vertex but the two poles has exactly one bundle of incoming edges and one bundle of outgoing ones, hence its corners may be classified into some polar ones (source or sink) and two lateral ones (left and right);

3. (any plane embedding of) the quadrangulation of the map can be endowed with an  $\alpha$ -orientation so that each vertex but the outer ones has 2 incoming edges, and outer vertices have one; hence the set of bipolar orientations of a plane non separable map  $M$  is endowed with a lattice structure inherited from the lattice structure of the  $\alpha$ -orientations of its quadrangulation;
4. the minimal element of this lattice is the unique bipolar orientation of  $M$  such that, for any distinct vertices  $u$  and  $v$  both incident to distinct faces  $f$  and  $g$ , the following configuration is forbidden:  $u$  on the right of  $f$  and source of  $g$ ,  $v$  sink of  $f$  and on the left of  $g$ , see Fig. 9(c). This pattern corresponds to a counterclockwise 4-cycle in the quadrangulation.

The constructions for general and cubic non separable planar maps are based on this minimal bipolar orientation. In the following, we denote by  $s$  and  $t$  respectively the source and the sink of the considered bipolar orientations, and say that a face is *generic* if it is *not* incident to the root edge. We use the usual convention that bipolar oriented maps are drawn in the plane with oriented paths going upwards, where  $s$  and  $t$  are outer vertices,  $s$  is at the bottom and  $t$  at the top of the figure, and with this convention we choose the embedding in which the root edge is the rightmost one.

### 3.3.1 Non separable cubic maps

The case of cubic maps (treated in [34], extended in [32]) is very constrained: since each vertex has degree 3, in any bipolar orientation, each non polar vertex has either one incoming edge and two outgoing ones, or the opposite. Hence each vertex (including the two poles) is either source or sink of exactly one generic face. Let  $M$  be such a rooted non separable cubic map endowed with its minimal bipolar orientation, and let us add an extra *bipolar edge* in each generic face of  $M$  (that is, an edge between its two poles). These extra edges realize a perfect matching of the vertices, hence the resulting planar map  $\overline{M}$  is 4-regular. As such,  $\overline{M}$  can be endowed with its minimal Eulerian orientation, and opened accordingly into a blossoming tree rooted at the sink  $t$ .

**Lemma 3.5.** *All bipolar edges belong to the resulting blossoming tree  $\mathcal{B}_M$ , and each vertex carries exactly one opening stem, immediately before the bipolar edge in clockwise order.*

*Proof.* This is proved inductively on the number of non polar vertices of  $M$ . The lemma is true in the smallest case (three parallel edges between  $s$  and  $t$ ). Let now  $M$  have at least two inner vertices; let  $f$  be the (only) generic face incident to  $s$ ,  $g$  the bounded non generic face, and  $u$  the sink of  $f$ . The vertex  $u$  may be incident to the outer face (and even equal to  $t$ ); these specific cases are illustrated in Fig. 11, and the generic case is illustrated in Fig. 10. We show that the bipolar edge  $(s, u)$  satisfies the lemma, and in each possible case we build smaller non separable cubic maps such that  $\mathcal{B}_M$  is obtained from their respective blossoming trees by grafting them together with the small subtree made of  $s$ ,  $u$  and their incident edges and stems.

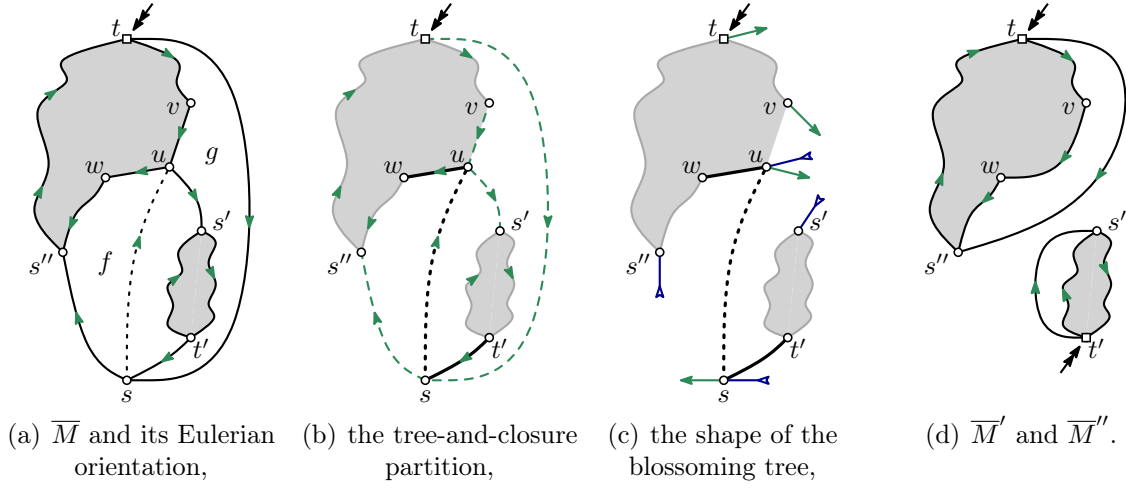


Figure 10: Generic case of non separable cubic maps.

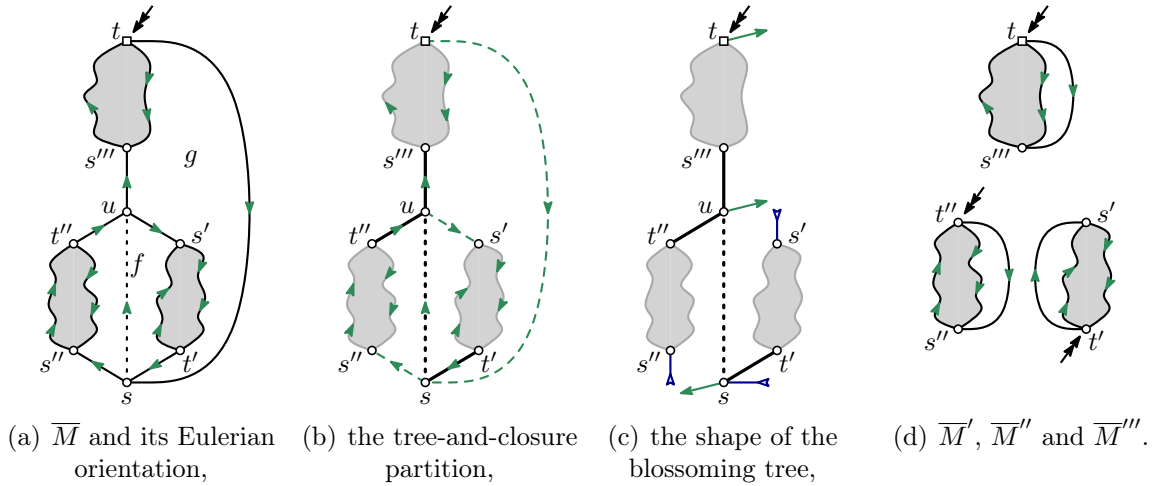


Figure 11: Specific *series parallel* case for non separable cubic maps.

Properties of the minimal Eulerian orientation of  $\overline{M}$  imply that the border of the outer face is a clockwise cycle, and that the edges incident to  $g$  are oriented clockwise or counterclockwise around  $g$  depending on whether they are incident to the outer face or not. Observe also that the two outer edges incident to  $s$  are oriented clockwise and since they are unnecessary to the accessibility of  $t$ , they belong to the closure. Therefore the bipolar edge  $(s, u)$ , oriented from  $s$  to  $u$ , belongs to the tree.

To get that an opening stem precedes the bipolar edge around  $u$ , the key point is to see that, due to the minimality of the underlying bipolar orientation, the face on the right of  $u$  is the non generic bounded face  $g$ , that is, its source and sink are  $s$  and  $t$ : otherwise, call  $\tilde{g}$  the face on the right of  $u$ ,  $v$  its source, and let  $\tilde{f}$  be the face on the left of  $v$  and  $\tilde{u}$  the sink of  $\tilde{f}$  ( $\tilde{f}$  and  $\tilde{u}$  may be equal to  $f$  and  $u$  respectively); then  $\tilde{u}$ ,  $v$ ,  $\tilde{f}$  and  $\tilde{g}$  would define exactly the forbidden pattern 9(c). This implies that the edge following the bipolar edge  $(s, u)$  in counterclockwise order around  $u$ , incident to both  $f$  and  $g$ , is necessarily outgoing, and hence belongs to the closure, which proves the lemma locally for  $s$  and  $u$ .

Now let  $s'$  and  $t'$  be the respective neighbours of  $u$  and  $s$  that are both incident to  $f$  and  $g$  – possibly equal to  $s$  and  $u$ . If not, deleting  $(u, s')$  and  $(t', s)$  disconnects  $M$  into two submaps; as these edges are both oriented clockwise around  $f$ ,  $(u, s')$  belongs to the closure and  $(t', s)$  belongs to the tree. Adding a root edge between  $s'$  and  $t'$  in the corresponding submap leads to a smaller non separable cubic map  $M'$ .

To end the construction in the generic case, let  $s''$  be the last neighbour of  $s$ , and let  $v$  and  $w$  be the two last neighbours of  $u$ , with  $v$  incident to  $g$  and  $w$  incident to  $f$ , see Fig. 10. Then  $(u, w)$  is a tree edge, while  $(s, s'')$  and  $(v, u)$  are closure edges. Let  $M''$  be obtained from  $M$  by removing  $s$  and  $u$  and their incident edges, and adding to the component of  $t$  and  $s''$  successively in the outer face a (closure) edge between  $v$  and  $w$ , and a (closure) root edge between  $t$  and  $s''$ . Then  $M''$  is a smaller non separable cubic map, and  $\mathcal{B}_M$  is obtained from  $\mathcal{B}_{M''}$  by grafting the subtree made from  $u$ ,  $s$  and  $\mathcal{B}_{M'}$  instead of the suitable stem of  $w$ .

Now, in the case where  $u$  is incident to the outer face, the situation between the face  $f$  and the outer face is similar to the one between the faces  $f$  and  $g$ , as illustrated in Fig. 11. Let  $s''$  and  $t''$  be the neighbours of  $s$  and  $u$  between these two faces, then  $(s, s'')$  belongs to the closure and, as soon as the two edges are distinct,  $(t'', t)$  belongs to the tree. Adding a root edge between  $s''$  and  $t''$  in the corresponding submap leads to a smaller non separable cubic map  $M''$ .

If moreover  $u \neq t$ , let  $s'''$  be its fourth neighbour; the edge  $(u, s''')$  (oriented towards  $s'''$ ) is incident to the two non generic faces, hence deleting the root edge and  $(u, s''')$  disconnects  $M$ , which implies in particular that  $(u, s''')$  belongs to the tree. In this case, let  $M'''$  denote the submap containing  $s'''$  and  $t$ , with an additional root edge between  $s'''$  and  $t$ :  $M$  is somehow a *series parallel* compound of three submaps  $M'$ ,  $M''$  and  $M'''$ , each possibly empty in degenerate cases, see Fig. 11.  $\square$

It is then quite clear that these balanced blossoming trees are exactly those of [34, 32], and can be described in a very simple manner. Non root opening stems are redundant since they immediately follow bipolar edges in counterclockwise order around vertices,

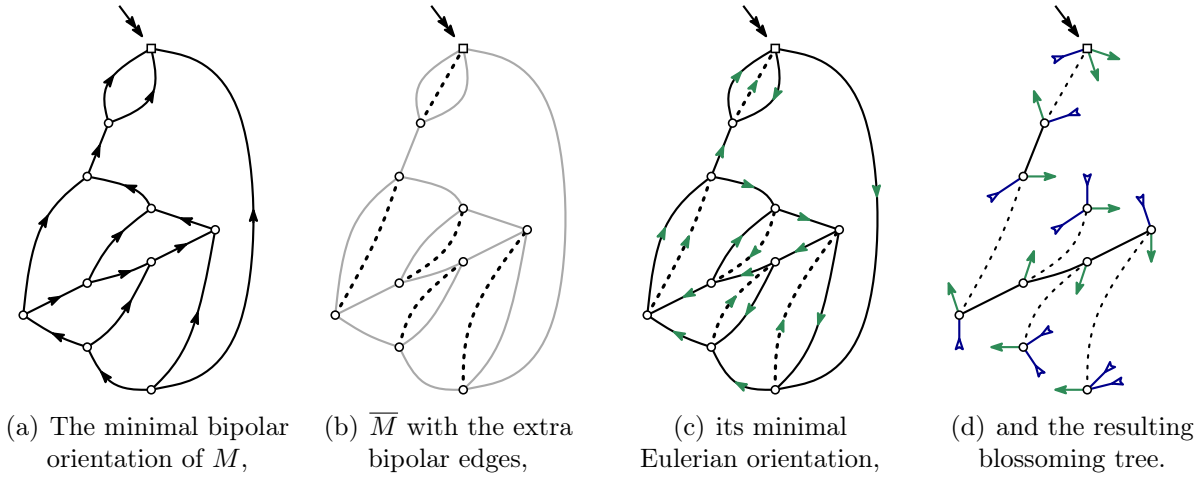


Figure 12: Complete example of a non separable cubic map  $M$ .

hence they can be removed. Then contracting bipolar edges yields a planted ternary tree (leaves of which are the closing stems). Conversely, let  $T$  be such a balanced blossoming *twin ternary* tree (that is, obtained by fairly splitting each node of a ternary tree in two *twin* nodes linked by a special *twinning edge*, with an additional opening stem on each node right before its twinning edge in clockwise order). For any subtree (a node, its stems and its descendants), let us call *free* the stems that are matched with stems not belonging to the subtree. An immediate counting shows that each subtree has one more closing stem than opening ones, hence one more free closing stem than free opening ones. Let us denote by  $s$  the node carrying the closing stem  $c$  corresponding to the (opening) root stem, and let  $u$  be its twin node;  $u$  is either its child or its parent. If  $u$  were the right child of  $s$ , its opening stem would be matched with the (only) closing stem of  $s$ . If  $u$  were its left child, the subtree of  $s$  would have at least two free closing stems since the opening stem of  $u$  would indeed be free, which prevents  $c$  from being matched with the root stem. Hence  $u$  is necessarily the parent of  $s$ , and as the opening stem of  $u$  is matched to a closing stem in the right subtree of  $s$ ,  $c$  is necessarily just before the opening stem of  $s$  in clockwise order around  $s$ . Hence  $s$  and  $u$  are exactly in the configuration described in the proof of Lemma 3.5.

Hence rooted planar non separable cubic maps with  $2n$  vertices are in one-to-one correspondence with balanced blossoming twin ternary trees with  $2n$  nodes. The number of planted ternary trees with  $n$  nodes is  $\frac{1}{2n+1} \binom{3n}{n}$ , each one leading to  $2^n$  distinct planted blossoming twin ternary trees. A fraction  $\frac{2}{2n+2}$  of them is balanced, leading to:

**Corollary 3.6.** *The number of rooted planar non separable cubic maps with  $2n$  vertices and  $3n$  edges is equal to:*

$$\frac{2^n}{(n+1)(2n+1)} \cdot \binom{3n}{n}.$$

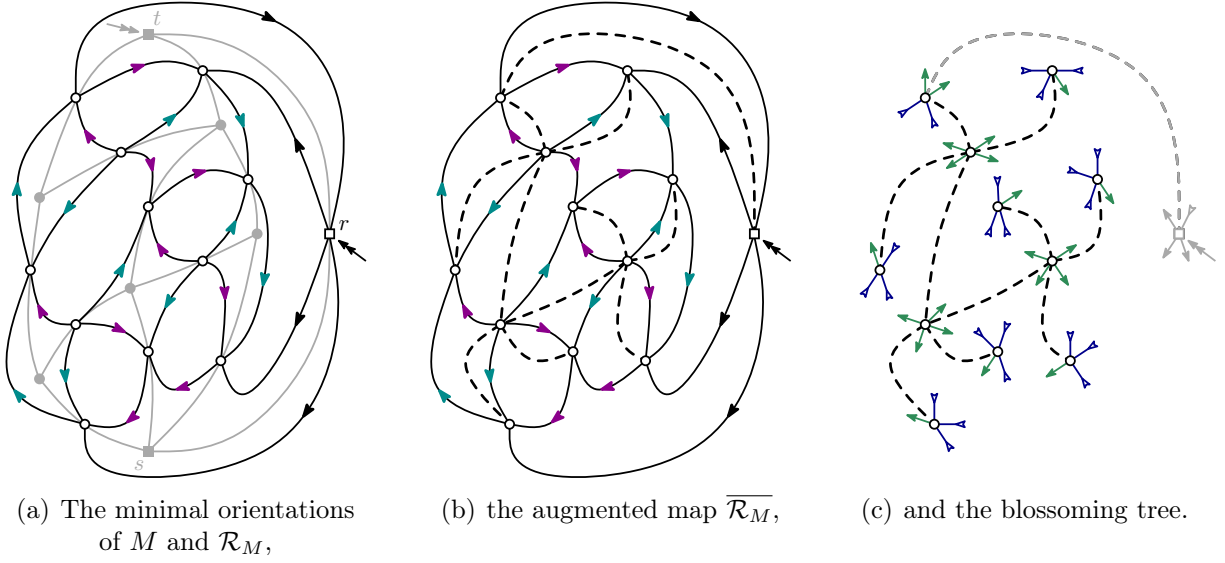


Figure 13: Complete example of a non separable planar map  $M$ .

### 3.3.2 General non separable maps

In the case of general planar non separable maps, the generic scheme is applied on (an extension of) their radial maps, as in Corollary 3.2. Let  $M$  be a rooted non separable map, and  $\mathcal{R}_M$  its radial map, rooted with the same root face as  $M$ , and root vertex  $r$  corresponding to the root edge of  $M$ . The existence of bipolar orientations for  $M$  is equivalent to the existence of orientations of  $\mathcal{R}_M$  with two clockwise edges per generic face; as for edges incident to  $r$ , we adopt the convention that the two outer ones are oriented clockwise, and the last two ones are outgoing for the root vertex. This orientation is said *minimal* if the corresponding bipolar orientation of  $M$  is itself minimal, see Fig. 13(a).

Given such an orientation, each generic face has two special corners (the origins of the two clockwise edges), and so does the face that corresponds to  $t$ . Let us add an extra edge in each such face between these two corners, and denote respectively  $\overline{\mathcal{R}_M}$  the resulting map and  $\mathcal{T}_M$  the map made exactly of these extra edges and their incident vertices, see Fig. 13(b). Then:

**Lemma 3.7** ([34]).  *$\mathcal{T}_M$  is a spanning tree of  $\overline{\mathcal{R}_M}$  if and only if the underlying orientation of  $\mathcal{R}_M$  is minimal.*

Hence in the minimal case,  $\mathcal{R}_M$  is actually a valid set of closure edges around  $\mathcal{T}_M$ , whatever accessible orientation is chosen for edges in  $\mathcal{T}_M$ . For instance, we may orient all edges towards  $r$ , or simply leave them unoriented (that is, oriented both ways in a 2-fractional orientation). Then  $(\mathcal{T}_M, \mathcal{R}_M)$  is the tree-and-closure partition of  $\overline{\mathcal{R}_M}$ .

The resulting (balanced) blossoming tree is such that each non root vertex is incident to 4 stems, and as many edges as opening stems – hence it has in-degree equal to 4. Considering closing stems as leaves, and after some surgery to remove the root vertex, we

get a planted ternary tree with one extra opening stem at each corner before an inner edge (clockwise around each vertex), see Fig. 13(c). Reciprocally, as shown in [34], the closure edges of such a balanced blossoming tree  $T$  form a 4-regular map  $R$  endowed with an orientation with 2 clockwise edges per face, which by Lemma 3.7 ensures that  $R$  is the radial map of a non separable map, endowed with its minimal orientation. Hence:

**Corollary 3.8.** *The number of rooted planar non separable maps with  $n$  edges is equal to:*

$$\frac{4}{(2n+1)(2n+2)} \cdot \binom{3n}{n}.$$

### 3.4 Simple triangulations and quadrangulations

Bijections between simple triangulations or quadrangulations and blossoming trees, as described in [33, 20], are special cases of the general bijection for  $d$ -angulations of girth  $d$  as explained in Sections 5 and 6, with a special emphasis in Subsection 6.3. In particular, the uniqueness part in Theorem 2.3 gives a more direct proof that the closure construction of [33, 20] for simple triangulations and quadrangulations of a  $p$ -gon is injective, while the existence part proves surjectivity without requiring a cardinality argument.

Besides, it is noteworthy that in such cases where the degree of faces are prescribed, closing stems are redundant; since the underlying orientation is regular, blossoming trees are trees with a fixed number of opening stems per vertex.

## 4 Plane bipolar orientations

Recall that a bipolar orientation of a map is an acyclic orientation of its edges with a single source (vertex without incoming edge) and a single sink (vertex without outgoing edge). A *plane bipolar orientation* is a corner-rooted map endowed with a bipolar orientation such that the root vertex of the map is the sink of the orientation and its source is the other extremity of the root edge, see Fig. 14(a). We emphasize that this section is devoted to the study of *all* plane bipolar orientations, as opposed to Section 3.3 which focuses only on maps endowed with their minimal bipolar orientation as a tool to enumerate non separable maps. Plane bipolar orientations have a nice enumerative formula:

**Theorem 4.1** (Baxter [2]). *For all positive integers  $i$  and  $j$ , the number  $\Theta_{ij}$  of plane bipolar orientations with  $i$  non-pole vertices and  $j$  generic faces is equal to:*

$$\Theta_{ij} = \frac{2(i+j)!(i+j+1)!(i+j+2)!}{i!(i+1)!(i+2)!j!(j+1)!(j+2)!}. \quad (1)$$

The first proof of this formula was given by Baxter [2]. His proof involves quite technical computation and relies on a “guess and check” approach. Since then, some bijective proofs of this result have been obtained in [21], [6] and [19]. Our generic scheme provides a new bijective proof.

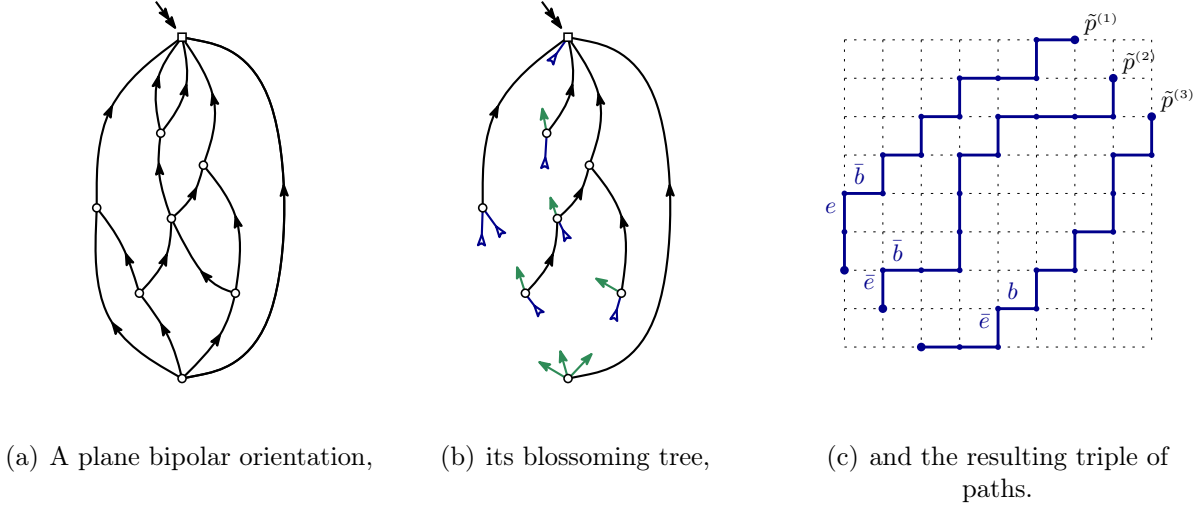


Figure 14: Example of the bijection for a bipolar orientation  $M$ .

## 4.1 Bijection with triple of paths

Let us consider paths on  $\mathbb{Z}^2$  made of right-steps  $(0, 1)$  and up-steps  $(1, 0)$ . A configuration of such paths is called *non-intersecting* if each vertex of  $\mathbb{Z}^2$  belongs to at most one path. For  $i, j \in \mathbb{N}$ , define the set  $\mathcal{P}_{i,j}$  of non-intersecting triple of paths  $(p^{(1)}, p^{(2)}, p^{(3)})$ , each made of exactly  $i$  right- and  $j$  up-steps and starting respectively at  $(-1, 1)$ ,  $(0, 0)$  and  $(1, -1)$  (and hence ending at  $(i-1, j+1)$ ,  $(i, j)$  and  $(i+1, j-1)$ ), see Fig. 14(c).

The rest of this section is devoted to the proof of the following theorem, from which a direct application of Lindström-Gessel-Viennot Lemma [22] yields the enumerative result of Baxter:

**Theorem 4.2.** *For all positive integers  $i$  and  $j$ , there exists a one-to-one constructive correspondence between the set of plane bipolar orientations with  $i$  generic faces and  $j$  non-pole vertices and the set  $\mathcal{P}_{i,j}$ .*

Other bijections between plane bipolar orientations and the set  $\mathcal{P}$  already appear in [21, 19]. It must nevertheless be emphasized that the bijection we obtained is different, thus providing a first example where our general scheme yields a new bijective construction. To prove the theorem, we start by applying the generic scheme to open a bipolar orientation into a blossoming tree, which is then encoded by a non-intersecting triple of paths.

## 4.2 From bipolar orientations to configurations of paths

Since any bipolar orientation is acyclic and its sink vertex  $t$  is accessible from any vertex, Corollary 2.4 can be applied to open it into a blossoming tree, rooted at the former outer corner of  $t$ . Let  $\mathcal{T}$  (resp.  $\mathcal{T}_{i,j}$ ) be the set of balanced blossoming trees obtained when opening a plane bipolar orientation (resp. with  $i$  generic faces and  $j$  non-pole vertices).



Let  $T$  be a rooted blossoming tree, we consider its contour word  $w$  on the alphabet  $\{e, \bar{e}, b, \bar{b}\}$ , where  $e$  and  $\bar{e}$  encode respectively the first and second exploration of an edge. Let furthermore define  $w^{(1)}$ ,  $w^{(2)}$  and  $w^{(3)}$  as the subwords of  $w$  obtained respectively by keeping only the letters  $e$  and  $\bar{b}$ ,  $\bar{e}$  and  $\bar{b}$ ,  $\bar{e}$  and  $b$ :

$$w^{(1)} = w_{|e, \bar{b}}, \quad w^{(2)} = w_{|\bar{e}, \bar{b}} \quad \text{and} \quad w^{(3)} = w_{|\bar{e}, b}.$$

**Claim 4.3.** *For  $T$  a blossoming tree of  $\mathcal{T}_{i,j}$ , the words  $w^{(1)}$ ,  $w^{(2)}$  and  $w^{(3)}$  have length  $\ell = i + j + 2$  and furthermore:*

$$w_1^{(1)} = e \text{ and } w_\ell^{(1)} = \bar{b}, \quad w_1^{(2)} = \bar{e} \text{ and } w_2^{(2)} = \bar{b}, \quad w_1^{(3)} = b \text{ and } w_\ell^{(3)} = \bar{e},$$

where  $w_k$  denotes the  $k$ -th letter of a word  $w$ .

For  $T \in \mathcal{T}$ , Claim 4.3 enables to define  $(\tilde{w}^{(1)}, \tilde{w}^{(2)}, \tilde{w}^{(3)})$  as:

$$w^{(1)} = e \tilde{w}^{(1)} \bar{b}, \quad w^{(2)} = \bar{e} \tilde{w}^{(2)} \bar{b} \quad \text{and} \quad w^{(3)} = b \tilde{w}^{(3)} \bar{e}. \quad (2)$$

Triple of words  $(\tilde{w}^{(1)}, \tilde{w}^{(2)}, \tilde{w}^{(3)})$  can be naturally represented by triple of up-right paths  $(\tilde{p}^{(1)}, \tilde{p}^{(2)}, \tilde{p}^{(3)})$ , with initial points  $(-1, 1)$ ,  $(0, 0)$  and  $(1, -1)$ , by replacing letters  $e$  or  $\bar{e}$  by up-steps and letters  $b$  and  $\bar{b}$  by right-steps. Let  $\Phi$  be the application that associates to each tree of  $\mathcal{T}$  the corresponding triple of paths, see Fig. 14. Observe that if  $M$  has  $i$  generic faces and  $j$  non pole vertices, then the corresponding blossoming tree  $T$  has  $i + 1$  pairs of opening-closing stems and  $j + 1$  edges, therefore each of the paths  $\tilde{p}^{(1)}$ ,  $\tilde{p}^{(2)}$  and  $\tilde{p}^{(3)}$  have exactly  $i$  right-steps and  $j$  up-steps.

**Proposition 4.4.** *Let  $T$  be an element of  $\mathcal{T}_{i,j}$ . Then its image by  $\Phi$  is non-intersecting, in other words it belongs to  $\mathcal{P}_{i,j}$ .*

This follows from:

**Lemma 4.5.** *A word  $w$  on the alphabet  $\{e, \bar{e}, b, \bar{b}\}$  is the contour word of an element of  $\mathcal{T}$  if and only if the five following conditions hold:*

- (1)  $w_1 = e$  and  $w_2 = b$ ;      (2)  $w_{|e, \bar{e}}$  is a Dyck word;      (3)  $w_{|b, \bar{b}}$  is a Dyck word;
- (4) for all  $i < j$ , if  $w_i = b$  and  $w_j = \bar{b}$ , there exists  $k \in [i, j]$  such that  $w_k = \bar{e}$ ;
- (5) for all  $1 < i < j$ , if  $w_i = e$  and  $w_j = \bar{e}$ , there exists  $k \in [i, j]$  such that  $w_k = \bar{b}$ .

*Proof.* Let  $T$  be an element of  $\mathcal{T}$  and let  $w$  be its contour word. For the first condition, observe that the root edge, oriented from  $s$  to  $t$ , belongs to the blossoming tree and is the first edge encountered in the contour of  $T$ , thus giving a first letter  $e$ . Moreover,  $s$  is a leaf of  $T$  that carries only opening stems and at least one, so  $w_2 = b$ . Condition 2 reflects the fact that  $T$  is a tree and Condition 3 that it is balanced.

Conditions 4 and 5 are proved by contradiction. If Condition 4 does not hold, there exists a factor of  $w$  of the form  $be \cdots e\bar{b}$  or  $b\bar{b}$ . It implies that a vertex is matched with one of

its descendants (possibly itself) in the closure, producing an oriented cycle, contradiction. Similarly, if Condition 5 does not hold, there exists a leaf of  $T$  (different from  $s$ ) which does not carry a closing stem. It contradicts the uniqueness of the source.

Reciprocally, the first three conditions imply that  $w$  is the contour word of a balanced blossoming tree  $T$  such that the first subtree of the root is reduced to one edge  $(t, s)$ , where  $s$  carries at least one opening stem. To ensure that  $w$  is the contour word of a tree in  $\mathcal{T}$ , it is enough to prove (a) that each vertex of  $T$  different from  $s$  has at least one ingoing edge and (b) that any opening stem is explored after the subtree of  $T$  rooted at the corresponding vertex (equivalently, that for any occurrence of  $b$  in  $w$ , all the occurrences of  $\bar{e}$  that appear after it encode edges that do not belong to the subtree rooted at the corresponding vertex).

Each node of  $T$  has at least one ingoing edge per child and Condition 4 ensures that each leaf of  $T$  carries at least one closing stem, hence (a) is satisfied. Consider an occurrence of  $b$  in  $w$  and the first occurrence of  $\bar{b}$  after it. Consider the first occurrence of  $\bar{e}$  after  $b$  (which precedes  $\bar{b}$  by Condition 4). The corresponding occurrence of  $e$  necessarily precedes  $b$  by Condition 5, hence there is no occurrences of  $e$  between  $b$  and  $\bar{e}$ , this is precisely (b).  $\square$

*Proof of Proposition 4.4.* Denote  $w_{\leq \ell}$  the prefix of length  $\ell$  of a word  $w$ , and  $|w|_x$  the number of occurrences of  $x$  in  $w$ . Condition 2 implies that:

$$|w_{\leq \ell}^{(1)}|_{\bar{b}} \leq |w_{\leq \ell}^{(2)}|_{\bar{b}} \quad \text{and} \quad |w_{\leq \ell}^{(1)}|_e \geq |w_{\leq \ell}^{(2)}|_{\bar{e}}.$$

Consequently, the corresponding paths  $(p^{(1)}, p^{(2)})$  starting at  $(0, 0)$  are such that  $p^{(1)}$  lies above and on the left of  $p^{(2)}$  with possible common vertices or edges. However the two paths share no vertical edge but the first one by Condition 5. Hence, after translating  $p^{(1)}$  by an up-step,  $p^{(1)}$  and  $p^{(2)}$  are non-intersecting and so are  $\tilde{p}^{(1)}$  and  $\tilde{p}^{(2)}$ .

We can prove similarly that  $(\tilde{p}^{(2)}, \tilde{p}^{(3)})$  is non-intersecting by observing first that  $p^{(2)}$  lies above and on the left of  $p^{(3)}$  thanks to Condition 3 of Lemma 4.5 and that  $p^{(2)}$  and  $p^{(3)}$  cannot share horizontal edge by Condition 4. Translating  $p^{(3)}$  by a right-step and deleting the appropriate steps of  $p^{(2)}$  and  $p^{(3)}$  yield the desired result.  $\square$

### 4.3 From configurations of paths to blossoming trees

Let  $p = (\tilde{p}^{(1)}, \tilde{p}^{(2)}, \tilde{p}^{(3)})$  be a configuration of paths in  $\mathcal{P}_{i,j}$  and  $(w^{(1)}, w^{(2)}, w^{(3)})$  the corresponding triple of words. Let us decompose  $w^{(1)}$  and  $w^{(3)}$  as a sequence of factors according respectively to the occurrences of  $\bar{b}$  and  $\bar{e}$ :

$$w^{(1)} = e \cdot w_{[1]}^{(1)} \cdot w_{[2]}^{(1)} \cdot \dots \cdot w_{[j+1]}^{(1)} \quad \text{and} \quad w^{(3)} = b \cdot w_{[1]}^{(3)} \cdot w_{[2]}^{(3)} \cdot \dots \cdot w_{[i+1]}^{(3)},$$

where each factor  $w_{[k]}^{(1)}$  (resp.  $w_{[k]}^{(3)}$ ) is of the form  $e^* \bar{b}$  (resp.  $b^* \bar{e}$ ). These factors are “bricks” used to reconstruct a compatible word  $w$ . The order in which those bricks are added is driven by  $w^{(2)}$ : let  $\bar{w}$  be obtained from  $w^{(2)}$  by replacing its  $k$ -th occurrence of  $\bar{b}$  by  $w_{[k]}^{(1)}$  and its  $k$ -th occurrence of  $\bar{e}$  by  $w_{[k]}^{(3)}$ , and define finally  $w = e b \bar{w}$ .

**Proposition 4.6.** *Let  $p$  be an element of  $\mathcal{P}_{i,j}$  and let  $w$  be the corresponding word as defined above. The word  $w$  is the unique word compatible with  $p$  that satisfies the five conditions of Lemma 4.5. In other words it is the contour word of the unique blossoming tree of  $\mathcal{T}$  compatible with  $p$ .*

*Proof.* First observe that  $w$  is compatible with  $p$  and that no other such word may satisfy the conditions of Lemma 4.5: Conditions 4 and 5 imply that the factors  $w_{[k]}^{(1)}$  and  $w_{[\ell]}^{(3)}$  have to be factors of  $w$ , and the order in which they appear is completely determined by  $w^{(2)}$ .

Let us now prove that  $w$  encodes indeed an element of  $\mathcal{T}$ , by applying Lemma 4.5. Condition 1 is clearly satisfied. Conditions 4 and 5 follow also easily from the definition of the decomposition in factors of  $w^{(1)}$  and  $w^{(3)}$ : observe for instance, for Condition 4, that any occurrence of  $b$  in  $w$  (but the first one) comes from a factor  $w_{[k]}^{(3)}$  of the form  $b^* \bar{e}$ . The first occurrence of  $b$  does not either raise a problem since  $w_1^{(2)} = \bar{e}$  and is hence replaced by  $w_{[1]}^{(3)}$ .

It remains to prove Conditions 2 and 3, namely that  $w|_{e, \bar{e}}$  and  $w|_{b, \bar{b}}$  are Dyck words. We only give the proof for  $w|_{e, \bar{e}}$ , since both proofs work along the same lines. From the construction of  $w$ , the number of occurrences of  $e$  and of  $\bar{e}$  in  $w$  are both equal to  $i + 1$ , hence it is enough to prove that  $|w_{\leq k}|_e \geq |w_{\leq k}|_{\bar{e}}$ , for all  $k$ . We consider the following decomposition of  $w$  into product of factors:

$$w = eb \cdot w_{[1]} \cdot w_{[2]} \cdot \dots \cdot w_{[i+j+2]},$$

where each of the  $w_{[k]}$  is equal to the corresponding factor of  $w^{(1)}$  or of  $w^{(3)}$ . It is then enough to check that for each  $1 \leq k \leq i + j + 2$ :

$$\left| eb \prod_{i=1}^k w_{[i]} \right|_e \geq \left| eb \prod_{i=1}^k w_{[i]} \right|_{\bar{e}},$$

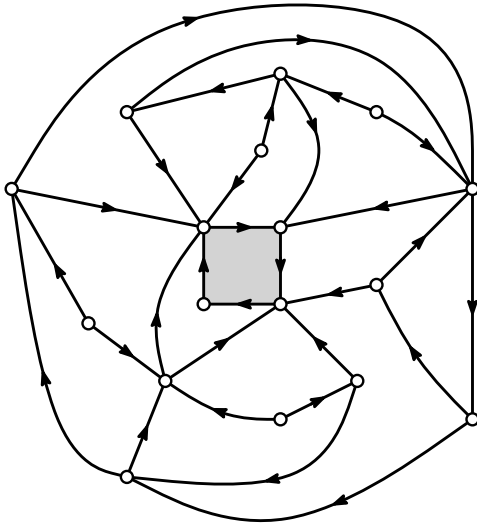
which can be rewritten as:

$$1 + \sum_{i=1}^{k_1} |w_{[i]}^{(1)}|_e \geq \sum_{i=1}^k |w_i^{(2)}|_{\bar{e}}, \quad \text{where } k_1 = |w_{\leq k}^{(2)}|_{\bar{b}}.$$

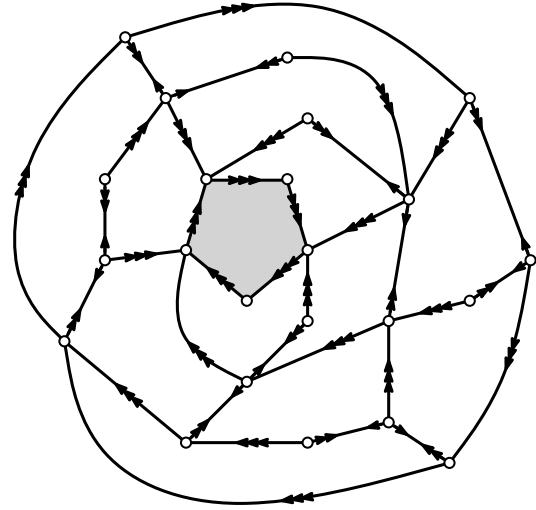
Let  $(x_2, y_2)$  be the point of  $p^{(2)}$  reached after  $k$  steps and let  $(x_2, y_1)$  be the point of  $p^{(1)}$  of abscissa  $x_2$  with minimal ordinate. By construction of  $w$ , the value of  $y_1$  is equal to the left-hand side of the above inequality, while  $y_2$  is equal to the right hand-side. Since  $p^{(1)}$  lies above  $p^{(2)}$ , we obtain the desired result.  $\square$

## 5 Blossoming trees for $d$ -angulations

The aim of this section is to generalize bijections previously obtained for simple triangulations [33] and simple quadrangulations [20], that is triangulations and quadrangulations without loops nor multiple edges. In other words, triangulations and quadrangulations in



(a) A simple quadrangulation endowed with its minimal 2-orientation.



(b) A pentagulation of girth 5 endowed with its minimal  $5/3$ -orientation.

Figure 15: Examples of  $\frac{d}{d-2}$  orientations for  $d$ -angulations of girth  $d$ .

which the contours of the faces are shortest cycles. More generally, the *girth* of a map is defined as the minimal length of its cycles. Obviously a  $d$ -angulation has girth at most  $d$  (except if it is a tree), hence simple triangulations and simple quadrangulations are exactly triangulations and quadrangulations with maximal girth. In the remaining sections, we aim at applying the general scheme to  $d$ -angulations of girth  $d$ , for any  $d \geq 3$ , and also to their following generalization. For any integers  $p \geq d \geq 3$ , define a  $d$ -angulation of a  $p$ -gon or a  $p$ -gonal  $d$ -angulation as a face-rooted plane map such that the contour of the root face is a simple cycle of length  $p$  and all non root-faces have degree  $d$ , see Fig. 16(a). We denote respectively  $\mathbf{M}_d$  and  $\mathbf{M}_{d,p}$  the set of  $d$ -angulations and  $p$ -gonal  $d$ -angulations of girth  $d$ , with distinct root and outer faces.

We do not use here the canonical plane embedding of face-rooted maps with the root face as the outer face. On the contrary, from now on, we consider only *face-rooted plane maps in which the outer face and root face are different*. This convention yields obviously equivalent enumerative byproducts but proves to fit better.

## 5.1 Orientations for $p$ -gonal $d$ -angulations

For any  $j, k \geq 0$ , a  $j/k$ -orientation of a face-rooted map is defined as a  $k$ -fractional orientation such that for each root vertex  $v$ ,  $\text{out}(v) = k$ , and  $\text{out}(v) = j$  otherwise (see Fig. 15(b)). Bernardi and Fusy show in [4] that the existence of  $\frac{d}{d-2}$ -orientations characterizes  $d$ -angulations of girth  $d$ , generalizing previous results obtained for triangulations [36] and quadrangulations [31]:

**Theorem 5.1** (Schnyder [36], Ossona de Mendez [31], Bernardi and Fusy [4]). *Let  $d \geq 3$*

and  $M$  be a face-rooted  $d$ -angulation; then  $M$  admits a  $\frac{d}{d-2}$ -orientation if and only if it has girth  $d$ . Moreover any such orientation is accessible.

Besides if  $d$  is even, all the flows in this orientation are even.

*Remark 1.* For  $d$  even, the parity of the flows implies that maps of  $\mathbf{M}_d$  can be endowed with  $\frac{d/2}{d/2-1}$ -orientations: for instance quadrangulations are naturally endowed with 2-orientations. For sake of conciseness we work here mainly with  $\frac{d}{d-2}$ -orientations and distinguish odd and even cases only when needed.

For  $p > d$ , a simple application of Euler formula proves that a  $p$ -gonal  $d$ -angulation cannot admit a  $\frac{d}{d-2}$ -orientation. The appropriate generalization is to define a (*pseudo-*) $\frac{d}{d-2}$ -orientation as a  $(d-2)$ -fractional orientation in which the contour of the root face is a circuit of saturated edges and  $\text{out}(v) = d$  for any non-root vertex  $v$ . Observe that for  $p = d$ , minimal pseudo- $\frac{d}{d-2}$ -orientations are minimal  $\frac{d}{d-2}$ -orientations. Pseudo- $\frac{d}{d-2}$ -orientations characterize  $p$ -gonal  $d$ -angulations of girth  $d$ :

**Proposition 5.2** (Proposition 19 of [4]). *Let  $p \geq d \geq 3$  be integers. A  $p$ -gonal  $d$ -angulation  $M$  admits a pseudo- $\frac{d}{d-2}$ -orientation if and only if it has girth  $d$ . In this case every pseudo- $\frac{d}{d-2}$ -orientation is accessible, and the sum of the outdegrees of the root vertices is equal to  $(d-2)p + (p-d)$ .*

Besides, there exists a unique minimal such pseudo- $\frac{d}{d-2}$ -orientation.

*Remark 2.* This result differs actually slightly from the result of [4]. Indeed, since we consider accessibility and minimality relatively to two different faces (the root face and the outer face), the map is allowed to be separated (*i.e.* there can exist a cycle of girth length that separates the root face and the outer face), which unifies the proof.

## 5.2 Bijection for $p$ -gonal $d$ -angulations

To adapt Theorem 2.3 to our context, we need to introduce the following family of planar maps akin to forests of trees. A *p-cyclic forest* is a face-rooted plane map with two faces, the root one and the outer one, such that the border of the root face is a simple cycle of length  $p$ . Observe that a cyclic forest is nothing else but a cycle of sequences of planted trees.

**Corollary 5.3.** *Let  $M = (V, E)$  be a plane face-rooted map with distinct root and outer faces and endowed with a minimal accessible orientation  $O$ . Then  $M$  admits a unique edge-partition  $(\mathcal{T}_M, \mathcal{C}_M)$  such that:*

- edges in  $\mathcal{T}_M$  form a spanning cyclic forest of  $M$  with same root-face as  $M$ , on which the restriction of  $O$  is accessible;
- edges in  $\mathcal{C}_M$  are saturated edges, and any of them turns clockwise around the unique cycle it forms with edges in  $\mathcal{T}_M$ .

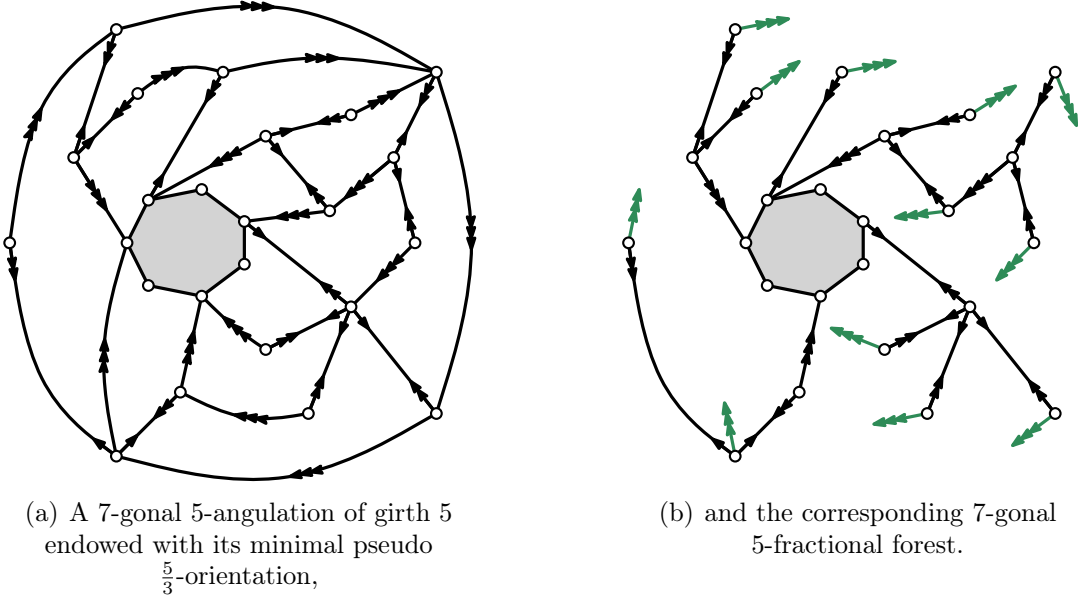


Figure 16: Example of the correspondence between  $p$ -gonal  $d$ -angulations of girth  $d$  and  $p$ -gonal  $d$ -fractional forests.

*Proof.* The map  $M$  admits such a unique partition of its edges if and only if  $\tilde{M}$  does, where  $\tilde{M}$  is the vertex-rooted map constructed from  $M$  by contracting its root face.  $\square$

The two following definitions describe the corresponding blossoming objects in the setting of  $p$ -gonal  $d$ -angulations (see Fig. 16(a)).

For any integers  $d \geq 3$  and  $0 \leq i < d - 2$ , a  $d$ -fractional tree of excess  $i$  is a planted blossoming tree endowed with an accessible  $(d - 2)$ -fractional orientation such that the root leaf has outdegree  $i$  and each non-root vertex has outdegree  $\text{out}(u) = d$ , where each opening stem contributes  $d - 2$ . The set of  $d$ -fractional trees of excess  $i$  (resp. with  $n$  vertices) is denoted by  $\mathsf{T}^{(i)}$  (resp.  $\mathsf{T}_n^{(i)}$ ).

A  $p$ -gonal  $d$ -fractional forest is a  $p$ -cyclic forest, the planted trees of which are  $d$ -fractional trees. The sum of their excesses is moreover required to be equal to  $p - d$ . Observe that such a forest is naturally endowed with a pseudo- $\frac{d}{d-2}$ -orientation. The set of  $p$ -gonal  $d$ -fractional forests (resp. with  $n$  vertices) is denoted by  $\tilde{\mathsf{F}}_d^p$  (resp.  $\tilde{\mathsf{F}}_d^p(n)$ ). The closure of a  $p$ -gonal  $d$ -fractional forest is the natural counterpart of the closure of blossoming map, in which the local closure of an opening stem creates a face of degree  $d$ .

The main theorem of this section is the following application of Corollary 2.4 to  $p$ -gonal  $d$ -angulations.

**Theorem 5.4.** *There exists a one-to-one constructive correspondence between  $p$ -gonal  $d$ -fractional forests with  $n$  vertices and  $p$ -gonal  $d$ -angulations of girth  $d$  with  $n$  non-root vertices.*

*Proof.* Corollary 5.3 and 2.4 entail that a  $p$ -gonal  $d$ -angulation endowed with its minimal  $\frac{d}{d-2}$ -orientation can uniquely be opened into a  $p$ -cyclic  $d$ -fractional forest with additional closing stems. Deleting them yields a  $p$ -cyclic  $d$ -fractional forest, from which they can be retrieved using the condition that non-root faces have degree  $d$ .

The only point to check is that the closure of a  $p$ -gonal  $d$ -fractional forest is indeed a  $p$ -gonal  $d$ -angulation, *i.e.* that the degree of the outer face of the full closure is  $d$ . A  $d$ -fractional tree of excess  $i$  with  $k$  vertices has  $\frac{2k+i}{d-2}$  opening stems. So a  $p$ -gonal  $d$ -fractional forest with  $n + p$  vertices has  $\frac{2n+p-d}{d-2}$  opening stems and its outer face has degree  $2n + p$ . Since each local closure reduces this degree by  $d - 2$ , it is exactly equal to  $d$  at the end of the closing process.

As mentioned in Section 2.3, the complexity of the opening procedure in the general case is quadratic. For  $p$ -gonal  $d$ -angulations, we were able to find a linear constructive algorithm. Its description is postponed to Section 6 for sake of clarity.  $\square$

### 5.3 Enumerative consequences

By Theorem 5.4, the enumeration of  $d$ -angulations of girth  $d$  boils down to the enumeration of  $d$ -fractional forests and the latter reduces to the counting of  $d$ -fractional trees of excess  $i$  for  $i \in \{0, \dots, d - 3\}$ . Let  $T_i$  be the generating series of  $\mathsf{T}^{(i)}$  according to the number of opening stems. The only way to generate and enumerate the elements of  $\mathsf{T}_n^{(i)}$  seems to require a recursive approach, described in this section. It comes as no surprise that the recursive scheme we obtain is essentially the same as the one that counts the  $d$ -regular mobiles of [4]. We found it nevertheless interesting to note that the same enumerative results can be obtained by our approach.

Following [4], for any positive integer  $j$ , let  $h_j$  be the polynomial in the variables  $t_1, t_2, \dots$  defined by:

$$h_j(t_1, t_2, \dots) := [x^j] \frac{1}{1 - \sum_{i>0} x^i t_i} = \sum_{r>0} \sum_{\substack{i_1, \dots, i_r > 0 \\ i_1 + \dots + i_r = j}} t_{i_1} \dots t_{i_r}.$$

In other terms,  $h_j$  is the generating function of compositions of  $j$  where the variable  $t_i$  keeps track of the number of parts of size  $i$ . Now, the generating series of the fractional trees of excess  $i$  can be obtained by a recursive decomposition. Cut the child of the root leaf to obtain a forest of trees (where one of the tree can be reduced to an opening stem) such that the sum of their excesses is equal to  $i + 2$ . In other words it can be written as a sequence  $(s_0, t_1, s_1, \dots, t_l, s_l)$ , where  $s_0, \dots, s_l$  are some sequences of trees with excess 0 (possibly reduced to a single vertex) and each of the  $t_i$ 's is a tree with positive excess. It yields the following system of equations:

$$T_i(x) = \frac{1}{1 - T_0} \cdot h_{i+2}\left(\frac{T_1}{1 - T_0}, \dots, \frac{T_{d-3}}{1 - T_0}, \frac{x}{1 - T_0}\right), \quad (3)$$

for  $0 \leq i \leq d - 3$  and  $T_i = 0$  otherwise. This set of equations characterizes  $T_0, T_1, \dots, T_{d-3}$  as formal power series. The constant coefficient of all these series is clearly equal to zero and the other coefficients can be computed recursively.

**Proposition 5.5** (Bernardi and Fusy [4]). *For  $p \geq d \geq 3$ , the generating function  $M_{d,p}(x)$  of corner-rooted  $p$ -gonal  $d$ -angulations of girth  $d$  with a marked outer face and counted according to the number of non-root faces is equal to:*

$$M_{d,p}(x) = x \left( \frac{1}{1 - T_0} \right)^p \cdot h_{p-d}^{(p)} \left( \frac{T_1}{1 - T_0}, \dots, \frac{T_{d-3}}{1 - T_0}, \frac{x}{1 - T_0} \right),$$

where  $h_j^{(p)}$  is defined in (4) and the series  $T_0, \dots, T_{d-3}$  are characterized by (3).

For  $p = d$ , this equation reduces to:

$$M_{d,d}(x) = x \left( \frac{1}{1 - T_0} \right)^d$$

*Proof.* By Theorem 5.4, we have:

$$M_{d,p}(x) = x F_{d,p}(x),$$

where  $F_{d,p}$  is the generating function of corner-rooted  $p$ -gonal  $d$ -fractional cyclic forests counted according to their number of opening stems. Erasing the edges of the root face of such a forest produces a  $p$ -tuple  $P := (P_1, \dots, P_p)$  of sequences of fractional trees such that the total sum of their excesses is equal to  $p - d$ .

For each  $1 \leq i \leq p$ , the sequence  $P_i$  can be written as  $P_i = (s_{i_0}, t_{i_1}, s_{i_1}, \dots, t_{i_l}, s_{i_l})$ , such that each of the  $s_j$  is a sequence made of trees of excess 0 (possibly empty) and each  $t_j$  is a tree of positive excess. This decomposition gives the following formula for  $F_{d,p}(x)$ :

$$F_{d,p}(x) = \left( \frac{1}{1 - T_0} \right)^p \cdot h_{p-d}^{(p)} \left( \frac{T_1}{1 - T_0}, \dots, \frac{T_{d-3}}{1 - T_0}, \frac{x}{1 - T_0} \right),$$

where for  $j \geq 0$ , the polynomial  $h_j^{(p)}$  is the generating function of  $p$ -tuples of compositions of integers, the sum of which is equal to  $j$ . For  $p = 0$ ,  $h_j^{(0)} = 1$ , for  $p = 1$  we retrieve the quantity  $h_j$  and more generally:

$$h_j^{(p)}(t_1, t_2, \dots) := [x^j] \frac{1}{(1 - \sum_{i>0} x^i t_i)^p}. \quad (4)$$

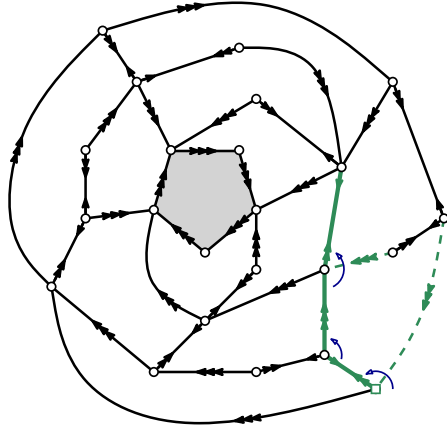
□

## 6 Fast opening of $p$ -gonal $d$ -angulations of girth $d$

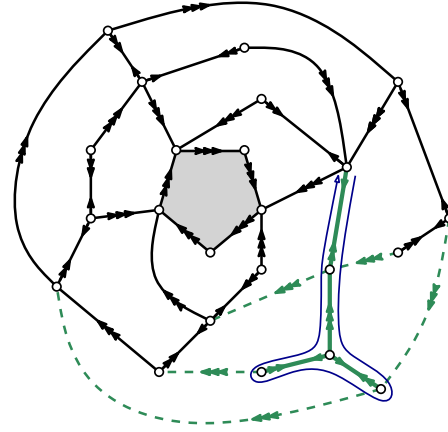
### 6.1 General description of the algorithm

This section is devoted to the description of a linear-time algorithm that associates to each element of  $\mathbf{M}_d$  its tree-and-closure partition or equivalently its  $d$ -fractional forest. Since all

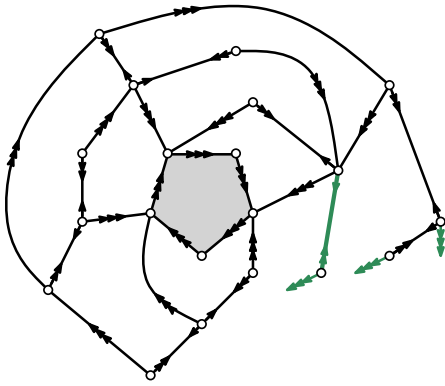




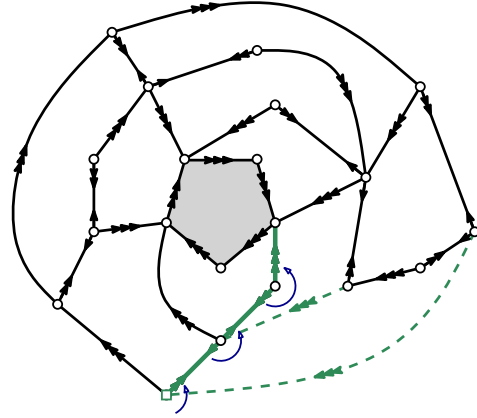
(a) Identification of a lineage path of length 3,



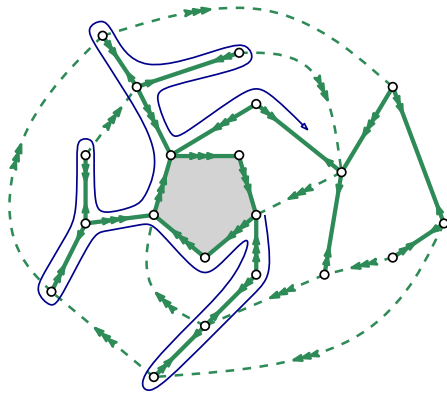
(b) the subtree  $\mathcal{B}_M^{(e_3)}$ ,



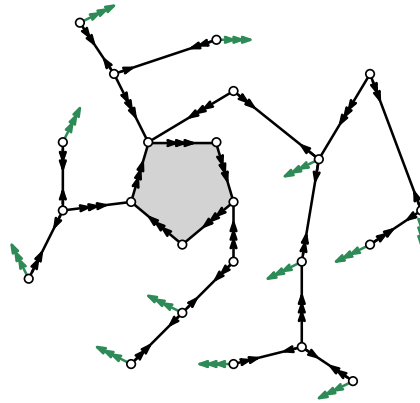
(c) replacing  $\mathcal{B}_M^{(e_3)}$  by  $t_1^{(5)}$ ,



(d) the lineage path reaches the root pentagon,



(e) the DFS identifies  $\mathcal{B}_{M/e_3}$  entirely,



(f) which enables to reconstruct  $\mathcal{B}_M$ .

Figure 17: Execution of the fast opening algorithm on the pentagulation of Fig.15(b).

steps extend easily to  $p$ -gonal  $d$ -angulations, we only give the proof in that setting for sake of conciseness.

Let  $M \in \mathbf{M}_d$ , recall that  $\mathcal{T}_M$ ,  $\mathcal{C}_M$  and  $\mathcal{B}_M$  denote respectively its sets of tree edges, of closure edges and the  $d$ -fractional forest associated to  $M$ . Since the root face of  $M$  is not its outer face, the construction of Proposition 2.5 cannot be applied directly to obtain  $\mathcal{B}_M$ . The general idea of our algorithm is however to use it iteratively to identify and cut subtrees of  $\mathcal{B}_M$ .

More precisely, for any edge  $e \in \mathcal{T}_M$ , let us denote by  $\mathcal{B}_M^{(e)}$  the (blossoming) subtree of  $\mathcal{B}_M$  planted at  $e$ , and define the *almost-total closure*  $M(e)$  of  $M$  relatively to  $e$  as the maximal partial closure of  $\mathcal{B}_M$  such that the border of the subtree  $\mathcal{B}_M^{(e)}$  (including both sides of  $e$ ) still lies on the outer face of  $M(e)$  (see Fig. 18(a)). Deleting  $\mathcal{B}_M^{(e)}$  yields a blossoming map with a canonically marked corner, denoted by  $M_{/e}$ . It is clear that  $M$  is the closure of the blossoming forest obtained by grafting  $\mathcal{B}_M^{(e)}$  on  $\mathcal{B}_{M_{/e}}$  in its marked corner. Uniqueness proved in Theorem 5.4 implies:

**Claim 6.1.** *For any map  $M \in \mathbf{M}_d$  and any edge  $e \in \mathcal{T}_M$ ,  $\mathcal{B}_M$  is the forest obtained by grafting  $\mathcal{B}_M^{(e)}$  in the marked corner of  $\mathcal{B}_{M_{/e}}$ .*

In other words, this result entails that computing the tree-and-closure partition of  $M$  can be achieved in two steps: first, identify  $M(e)$  for some tree-edge  $e$ , and then, compute the tree-and-closure partition of  $M_{/e}$ . To deal with the first point, observe that an adapted version of the depth-first search process described in Section 2.3 enables to compute  $M(e)$  in the case where  $e$  is an outer edge. It is stated in the following proposition, that loosens the conditions in which Proposition 2.5 can be applied. This proposition is illustrated in Fig.17(b) and its proof follows easily from the proof of Proposition 2.5.

**Proposition 6.2.** *Let  $M$  be a blossoming map, and  $(u, v)$  be one of its outer edges, with  $u$  preceding  $v$  in clockwise order around  $M$ . If  $(u, v)$  belongs to  $\mathcal{T}_M$  and  $u$  is the parent of  $v$ , then the construction described in Proposition 2.5 can be applied to identify  $\mathcal{B}_M^{(u, v)}$ , starting at vertex  $u$  and edge  $(u, v)$ , and stopping when returning at  $u$ .*

The purpose of Section 6.2 is then to identify such outer tree edges. This operation should then be iterated to obtain  $\mathcal{B}_{M_{/e}}$ . But since the closure of  $M_{/e}$  is not necessarily a  $d$ -angulation (this is indeed only the case if  $e$  is a saturated edge), some extra care has to be taken to continue the algorithm. Section 6.4 describes in full details the iteration step, via a slightly larger  $d$ -angulation. The case of triangulations and quadrangulations is somewhat simpler since all the edges are saturated and is treated separately in Section 6.3.

## 6.2 Opening along a lineage path from the outer face

Given a map  $M$ , Proposition 2.5 can be applied as soon as an outer tree edge of  $M$  (with the correct parent-child orientation) is identified. Unfortunately, such an edge does not necessarily exist. Nevertheless, a partial opening of  $M$  with the above property (and even

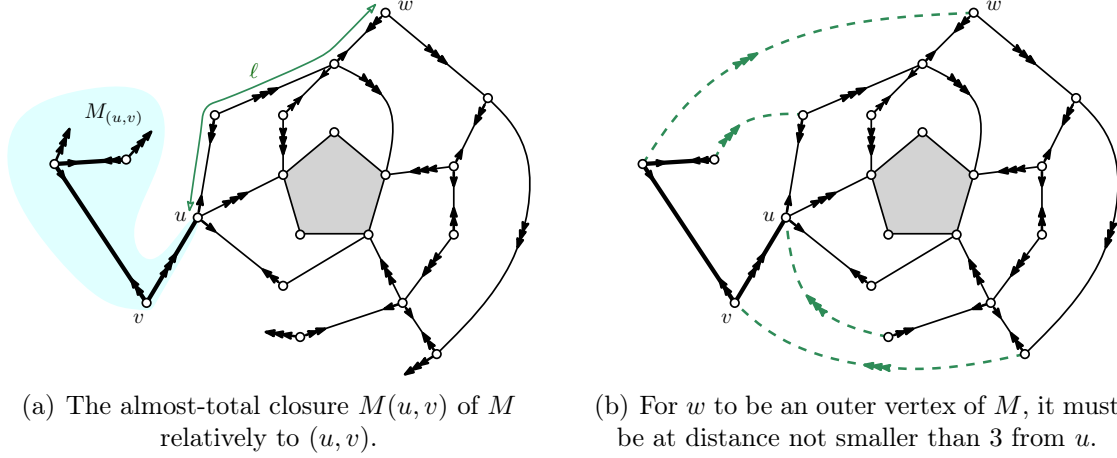


Figure 18: Illustration of Lemma 6.4 for a pentagulation.

with the stronger requirement that a lineage path of  $\mathcal{B}_M$  of length  $d/2$  lies on the outer face) can be constructed in linear time. We need the following technical result, see Fig. 18:

**Definition 6.3.** *The clockwise distance between two outer vertices  $u$  and  $w$  of a blossoming map is the length of the clockwise path from  $u$  to  $w$  along the border of its outer face.*

**Lemma 6.4.** *Let  $M \in \mathbf{M}_d$  and  $u, v$  in  $M$  such that  $v$  is a child of  $u$  in  $\mathcal{B}_M$ . Let  $w$  be an outer vertex of  $M$  that does not belong to  $\mathcal{B}_M^{(u,v)}$ . Then, the clockwise distance from  $u$  to  $w$  in  $M(u, v)$  cannot be smaller than  $\lceil d/2 \rceil$ .*

*Proof.* Let  $T = \mathcal{B}_M^{(u,v)}$ ,  $n$  be its number of edges, and  $0 \leq i \leq d-3$  be the flow from  $u$  to  $v$ . A simple combinatorial argument yields that the number of stems in  $T$  is equal to  $(2n + i)/(d-2)$ . It implies that performing all the local closures of stems in  $T$  requires  $2n + i + 1$  sides of edges (each stem needs  $d-1$  sides of edges, and each new created edge but the last one can be used in a further closure).

Now the number of sides of edges of  $T$  is equal to  $2n$ , but not all of them can be used in local closures involving outgoing stems of  $T$ . Indeed the first stem explored during a DFS contour of  $T$  is discovered after at least  $\lfloor (d-i-1)/2 \rfloor$  steps. Hence the number of sides of edges available for closures in  $T$  is equal to  $2n - \lfloor (d-i-1)/2 \rfloor$ . It follows that closing the stems of  $T$  requires at least  $\lfloor (d+1)/2 \rfloor = \lceil d/2 \rceil$  additional sides of edges; hence outer vertices of  $M(u, v)$  at clockwise distance less than  $\lceil d/2 \rceil$  from  $u$  are separated from the outer face of  $M$  by at least one closure edge.  $\square$

This lemma implies that if  $(u, v)$  is a tree-edge and  $u$  is the parent of  $v$ , then the outer face cannot lie on the left of  $(v, u)$ . In particular:

**Corollary 6.5.** *Let  $M \in \mathbf{M}_d$ . Any outer clockwise saturated edge of  $M$  (and there exists at least one of them) is a closure edge, and other outer edges (if any) have the outer face on their right when walking from a child to its parent.*

We now describe the construction of a partial opening of  $M$ , which admits an outer lineage path, as illustrated on Figures 17(a) and 17(d):

**Proposition 6.6.** *Let  $M \in \mathbf{M}_d$  and  $e_0 = (u_0, v_0)$  be one of its outer closure edges. Initialize  $C = \{e_0\}$ ,  $e = e_0$  and  $i = 0$ . Then repeat until either  $i = \lceil d/2 \rceil$  or  $v_i$  is a root vertex:*

*Update  $e$  to the next edge  $(w, v_i)$  around  $v_i$  in counterclockwise direction.  
If  $e$  is saturated from  $w$  to  $v_i$  then add it to  $C$ ,  
else increment  $i$ , set  $v_i$  to  $w$  and  $e_i$  to  $(v_i, v_{i-1})$ .*

*Let  $\ell$  be the final value of  $i$ . Then  $C$  is included in  $\mathcal{C}_M$  and the path  $(v_0, \dots, v_\ell)$  is a lineage path in  $\mathcal{T}_M$  oriented towards the root. Furthermore each  $v_i$  is an outer vertex of  $M \setminus C$ .*

*Proof.* We prove the result by induction on the number of iterations. Assume the conclusion of the proposition is true after  $k$  steps of the algorithm and let  $i_k$  be the corresponding value of  $i$ . Since  $v_0$  is an outer vertex of  $M$ , at clockwise distance  $i_k < d/2$  from  $v_{i_k}$  around  $M \setminus C$ ,  $w$  cannot be the child of  $v_{i_k}$  by Lemma 6.4. Hence if  $e$  is saturated towards  $v_i$ , it implies that  $e$  is a closure edge; otherwise  $w$  is the parent of  $v_{i_k}$ . Moreover, from the construction,  $w$  is necessarily an outer vertex of  $M \setminus C$ .  $\square$

**Definition 6.7.** *Let  $M \in \mathbf{M}_d$ ; any outer closure edge  $e_0$  of  $M$  defines a tree edge  $e_\ell$  by Proposition 6.6. The set of subtrees  $\mathcal{B}_M^{(e_\ell)}$  is denoted  $\mathcal{S}_M$ .*

Let us end this section with a few comments. The subtree  $\mathcal{B}_M^{(e_\ell)}$  can be identified by applying the construction of Proposition 6.2. We could in fact have applied this proposition to the edge  $e_1$  but relying on this whole lineage path, we can identify a much bigger subtree. This is actually a key point for the next step described in Section 6.4.

*Remark 3.* If the lineage path reaches the root face, the depth-first search algorithm can be continued so as to identify  $\mathcal{B}_M$  entirely, as in Figure 17(e): indeed in this case we are precisely in the setting of Proposition 2.5.

### 6.3 Triangulations and quadrangulations

Simple triangulations and simple quadrangulations constitute a much simpler case than general  $d$ -angulations of girth  $d$ , since in the minimal  $\frac{d}{d-2}$ -orientation canonically associated to them all the edges are saturated.

Given a  $d$ -angulation  $M$  of girth  $d$  with  $d = 3$  or  $d = 4$ , Proposition 6.6 identifies two tree-edges  $e_1$  and  $e_2$ . Now Proposition 6.2 enables to compute  $M(e_2)$  and to decompose it into  $\mathcal{B}_M^{(e_2)}$  and  $M_{/e_2}$ . Since all the edges are saturated, the closure of  $M_{/e_2}$  is itself a  $d$ -angulation, strictly smaller than  $M$  and we can iterate the operations on this new map.

At each iteration the number of edges of the map decreases, the sequence of maps reaching eventually the trivial map reduced to a single cycle. Following the sequence backwards provides a recursive construction of the tree-and-closure partition of the desired map.

## 6.4 How to graft small subtrees

This section is dedicated to the iteration and the proof of the termination of the algorithm in the general case of  $d$ -angulations of girth  $d$  for  $d \geq 5$ . Roughly speaking, we want to iterate the construction of Section 6.2 on the closure of  $M_{/e_\ell}$ , but, as mentioned in Section 6.1, this map is not necessarily a  $d$ -angulation. This can happen if the edge  $e_\ell$  is not saturated, so that its deletion creates a vertex  $u$  of outdegree strictly less than  $d$ . To circumvent this problem, we graft at  $u$  the smallest  $d$ -fractional subtree of appropriate excess, according to an ad-hoc notion of order on  $d$ -fractional trees, as described below.

A natural partial order on blossoming trees is the order induced by sizes: a tree  $t_1$  is declared to be smaller than a tree  $t_2$  if  $t_1$  has strictly less edges than  $t_2$ . But this order has to be slightly refined so as to define unambiguously a unique minimal element  $t_i^{(d)}$  in each  $\mathsf{T}^{(i)}$ . Precisely:

- We adopt the convention that the empty tree has excess 0, hence it is equal to  $t_0^{(d)}$ .
- For  $d$  even or  $i$  odd, there exists also a unique minimal element  $t_i^{(d)}$  in  $\mathsf{T}^{(i)}$ , the one made of a path of length  $(d - 2 - i)/2$  and one stem.
- For  $d$  odd and  $i$  even, exactly two trees have minimal size, both made of a path of length  $d - 2 - i/2$  with two stems; consider their contour words on  $\{b, e, \bar{e}\}$ : they are respectively equal to  $w_1 = e^p b e^q b \bar{e}^{(p+q)}$  and  $w_2 = e^{p+q} b \bar{e}^q b \bar{e}^p$ , with  $p = \frac{1}{2}(d - i - 1)$  and  $q = \frac{1}{2}(d - 3)$ . We refine the ordering on  $\mathsf{T}^{(i)}$  by defining the minimal element  $t_i^{(d)}$  as the one with contour word  $w_1$ .

This notion of ordering extends naturally to  $d$ -fractional forests:  $F_1 < F_2$  if and only if  $F_1$  can be obtained from  $F_2$  by replacing a sequence of subtrees by smaller ones. The maps in  $\mathsf{M}_d$  inherit the ordering of their blossoming forests. The unique minimal map of  $\mathsf{M}_d$  is the map reduced to a simple cycle of length  $d$ .

**Proposition 6.8.** *Let  $M \in \mathsf{M}_d$ ,  $T \in \mathcal{S}_M$  and assume that the excess of  $T$  is equal to  $i$ . Then  $T$  is not equal to  $t_i^{(d)}$ .*

*Proof.* The case  $i$  equal to 0 is clear. The construction of Proposition 6.6 implies that any tree of  $\mathcal{S}_M$  has at least  $d/2$  edges before the first stem in its contour word (except if  $v_\ell$  is a root vertex, in which case the excess  $i$  is equal to 0). Since  $t_i^{(d)}$  has less than  $d/2$  edges for  $d$  even or  $i$  odd, this proves the result in those cases. For  $d$  odd and  $i$  even, our particular choice for  $t_i^{(d)}$  enables also to conclude.  $\square$

## 6.5 The opening algorithm

Let  $M$  in  $\mathsf{M}_d$ , define a sequence  $(M_k)$  of maps of  $\mathsf{M}_d$ , where  $M_0 = M$  and  $M_{k+1}$  is obtained from  $M_k$  in the following way:

- Identify a tree  $T_k$  in  $\mathcal{S}_{M_k}$  by Propositions 6.6 and 6.2 and construct the corresponding almost-total closure  $M'_k$ ;

- Replace  $T_k$  by the adequate  $t_i^{(d)}$  in  $M'_k$  to obtain  $M'_{k+1}$ ;
- Let  $M_{k+1}$  be the closure of  $M'_{k+1}$ .

By Proposition 6.8, this sequence is decreasing and hence converges eventually to the trivial map. In fact, following Remark 3, the construction of this sequence can be stopped as soon as the lineage path identified by Proposition 6.6 reaches the root face. Following the sequence backwards provides a recursive construction of the tree-and-closure partition of  $M$ . A complete run of the algorithm for the pentagulation of Fig.15(b).

**Proposition 6.9.** *The above algorithm can be implemented in time linear in the size of  $M$ .*

*Proof.* First, observe that the length of the sequence  $(M_k)$  is bounded by  $n/2 + 2n/(d-1)$ . Indeed, for each  $k$ ,  $M_k$  has at least two edges less than  $M_{k-1}$ , except in the case where  $t_i^{(d)}$  is grafted instead of the other tree in  $\mathcal{T}^{(i)}$  of minimal size. The latter case happens at most  $2n/(d-1)$  times, since any edge appears at most once in such a subtree.

The cost of the  $k$ -th step can be decomposed in two parts:

- the part  $t(k)$  corresponding to the exploration of the subtree  $T_k$ , including the computation of the lineage path,
- the part  $c(k)$  associated to the handling of closure edges encountered during the exploration of the lineage path.

The number of edges in  $T_k$  is the sum of the size of the adequate  $t_i^{(d)}$ , bounded by  $d$ , and the size difference between  $M_{k+1}$  and  $M_k$ . Hence, the sum over  $k$  of  $t(k)$  is linear in  $n$ . To deal with  $c(k)$ , let us first observe that a bundle of closure edges that end at a same vertex of the lineage path in  $M_k$  also end at a same vertex in  $M_{k+1}$ . Hence, assume the map is encoded in such a way that closure edges with the same head are stored in a linked list, then the cost  $c(k)$  is linear in the number of explored bundles of closure edges and is hence bounded by  $d$ .  $\square$

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